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CONTENTS

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SPECIAL TRAIN ALGEBRAS

By I. M. H. ETHERINGTON (*Edinburgh*)

[Received 2 July 1940]

IN a previous paper (4) I have defined three kinds of non-associative linear algebras, namely, *baric algebras*, *train algebras*, and *special train algebras*, which appear in the symbolism of genetics. Train algebras and special train algebras are by definition baric algebras; and I have asserted that special train algebras are train algebras. This paper supplies the proof.

The commutative law of multiplication is assumed in § 3 (not in §§ 1, 2); and the theorem is proved on this assumption in § 4, the extension to non-commutative algebras being indicated briefly. Methods of obtaining the principal and plenary train roots of a special train algebra are shown by an example in § 5.

1. Powers of a non-associative linear algebra

I have defined elsewhere (3) the *shape* s, *degree* δ, and *altitude* α of a non-associative product or power. (In the case of a power, the *shape* is the same as its *index*.) Of these δ, α are ordinary integers, δ being simply the number of factors; while s is an integer in an arithmetic with non-associative addition.

By the sth *power* of a linear algebra A, written A^s, is meant the set of all linear combinations of products having the shape s formed from elements of A. By the δth *involute* of A, written $\mathsf{A}^{[\delta]}$, is meant the sum of all powers of A of degree δ; in other words, the set of all linear combinations of products of δ elements of A.* By the αth *generation* of A, written $\mathsf{A}^{(\alpha)}$, is meant the sum of all powers of A of altitude α; in other words, the set of all linear combinations of products of altitude α formed from elements of A. Note that

$$\mathsf{A} = \mathsf{A}^{[1]} = \mathsf{A}^{(0)}.$$

By considering how products of a given degree or altitude can be formed from those of lower degree or altitude, we find that

$$\mathsf{A}^{[\delta]} = \mathsf{A}\mathsf{A}^{[\delta-1]} + \mathsf{A}^{[2]}\mathsf{A}^{[\delta-2]} + \ldots + \mathsf{A}^{[\delta-1]}\mathsf{A}, \tag{1.1}$$

$$\mathsf{A}^{(\alpha)} = \mathsf{A}\mathsf{A}^{(\alpha-1)} + \mathsf{A}^{(\alpha-1)}\mathsf{A}, \tag{1.2}$$

* Cf. (8), 111.

equations which can obviously be simplified when multiplication is commutative.

By grouping its factors a non-associative product of given shape can always be regarded as a product of simpler shape, fewer factors, lower altitude. Consequently

$$\mathsf{A}^{[\delta]} \leqslant \mathsf{A}^{[\delta-1]} \leqslant \ldots \leqslant \mathsf{A}^{[2]} \leqslant \mathsf{A}, \tag{1.3}$$

$$\mathsf{A}^{(\alpha)} \leqslant \mathsf{A}^{(\alpha-1)} \leqslant \ldots \leqslant \mathsf{A}^{(1)} \leqslant \mathsf{A}. \tag{1.4}$$

Since evidently

$$\mathsf{AA}^{[\delta]} \leqslant \mathsf{A}^{[\delta]}, \quad \mathsf{A}^{[\delta]}\mathsf{A} \leqslant \mathsf{A}^{[\delta]}, \quad \mathsf{AA}^{(\alpha)} \leqslant \mathsf{A}^{(\alpha)}, \quad \mathsf{A}^{(\alpha)}\mathsf{A} \leqslant \mathsf{A}^{(\alpha)}, \tag{1.5}$$

it follows that any involute or generation of A is an invariant subalgebra of A. On the other hand, a power of A, A^s, is not necessarily an algebra, since it may not be closed as regards multiplication.

2. Nilpotent algebras

A linear algebra A will be called *nilpotent of degree* δ if $\mathsf{A}^{[\delta]} = 0$, $\mathsf{A}^{[\delta-1]} \neq 0$; and *nilpotent of altitude* α if $\mathsf{A}^{(\alpha)} = 0$, $\mathsf{A}^{(\alpha-1)} \neq 0$. (Other writers use *index* in place of *degree* in this context.)

In the first case, any product of δ or more elements vanishes. Consider a product of altitude $\alpha' = \delta - 1$. Its degree δ' satisfies $\delta' \geqslant \alpha'+1$, that is, $\delta' \geqslant \delta$.* Hence this product vanishes, and thus, *if* A *is nilpotent of degree* δ, *it is nilpotent of altitude* $\delta-1$ *at most.*

In the second case, any product in A of altitude α vanishes. Consider a product of degree $\delta' = 2^\alpha$. Its altitude α' satisfies $\alpha' \geqslant \log_2 \delta'$, that is, $\alpha' \geqslant \alpha$.* Hence this product vanishes; and thus, *if* A *is nilpotent of altitude* α, *it is nilpotent of degree* 2^α *at most.*

Wedderburn† has stated that, if A is nilpotent of degree δ, then

$$\mathsf{A}^{[\delta-1]} < \mathsf{A}^{[\delta-2]} < \ldots < \mathsf{A}^{[2]} < \mathsf{A};$$

but‡ his proof is based on the incorrect equation§

$$\mathsf{A}^{[\delta]} = \mathsf{AA}^{[\delta-1]} + \mathsf{A}^{[\delta-1]}\mathsf{A}. \tag{2.1}$$

The same method of proof, however, using (1.2) instead of (2.1), yields the theorem: *If* A *is nilpotent of altitude* α, *then*

$$\mathsf{A}^{(\alpha-1)} < \mathsf{A}^{(\alpha-2)} < \ldots < \mathsf{A}^{(1)} < \mathsf{A}. \tag{2.2}$$

3. Canonical form of a commutative special train algebra

Let X be a special train algebra with commutative multiplication. The definition|| contains three postulates:

* See **(3)** (14). † **(8)** 111. ‡ Cf. **(6)**, § 4, footnote.
§ Cf. equation (1.1) here. || **(4)**, 246.

(1) X *is a baric algebra.* Hence* we can provide by a suitable linear transformation that one base element of X, say A, shall be of unit weight, and the rest, say u^σ ($\sigma = 1,\ldots, n$), of zero weight. We shall suppose this done. The u^σ form the basis of the nil sub-algebra U, which we know is an invariant sub-algebra of X; and the multiplication table of X takes the form

$$A^2 = A+(u), \qquad Au^\sigma = (u), \qquad u^\sigma u^\tau = (u), \tag{3.1}$$

where the u's in brackets denote unspecified linear combinations of the u^σ.

(2) U *is nilpotent of a certain degree.* As shown in § 2, this is equivalent to saying that U is nilpotent of a certain altitude, say α. Then we know that

$$0 = \mathsf{U}^{(\alpha)} < \mathsf{U}^{(\alpha-1)} < \mathsf{U}^{(\alpha-2)} < \ldots < \mathsf{U}^{(1)} < \mathsf{U}. \tag{3.2}$$

Consequently, by an appropriate linear transformation of the u^σ, we can separate the basis of U into sets of, let us say, u_0-elements, u_1-elements, u_2-elements, etc., such that the u_0's belong to U but not to $\mathsf{U}^{(1)}$, the u_1's belong to $\mathsf{U}^{(1)}$ but not to $\mathsf{U}^{(2)}$, the u_2's belong to $\mathsf{U}^{(2)}$ but not to $\mathsf{U}^{(3)}$, and so on. Since (cf. (1.2)) $\mathsf{U}\mathsf{U}^{(\theta)} = \mathsf{U}^{(\theta+1)}$, the multiplication table of U will take the form

$$\begin{array}{c|cccccc}
 & (u_0) & (u_1) & (u_2) & \cdots & (u_{\alpha-2}) & (u_{\alpha-1}) \\
\hline
(u_0) & (u_1, u_2,\ldots) & (u_2, u_3,\ldots) & (u_3,\ldots) & \cdots & (u_{\alpha-1}) & 0 \\
(u_1) & & (u_2, u_3,\ldots) & (u_3,\ldots) & \cdots & (u_{\alpha-1}) & 0 \\
(u_2) & & & (u_3,\ldots) & \cdots & (u_{\alpha-1}) & 0 \\
\cdot & & & & & \cdot & \cdot \\
\cdot & & & & & \cdot & \cdot \\
\cdot & & & & & \cdot & \cdot
\end{array} \tag{3.3}$$

That is to say, the product of any two u_0-elements belongs to $\mathsf{U}^{(1)}$ and hence may involve u_1-elements, u_2-elements, etc., but not u_0-elements; the product of a u_0 and a u_1 may involve u_2's, u_3's, etc., but not u_0's or u_1's; and so on.

(3) $\mathsf{U}^{(1)}, \mathsf{U}^{(2)},\ldots, \mathsf{U}^{(\alpha-1)}$ *are invariant sub-algebras of* X; so, of course, is U. This gives us the further information that

$$A(u_0) = (u_0, u_1,\ldots), \qquad A(u_1) = (u_1, u_2,\ldots), \qquad A(u_2) = (u_2, u_3,\ldots),\ldots. \tag{3.4}$$

* (5), Theorem II.

That is to say, the product of A and any u_θ may involve the u_θ's, $u_{\theta+1}$'s, $u_{\theta+2}$'s,..., but not the $u_{\theta-1}$'s, $u_{\theta-2}$'s,... .

Consider now each u_θ set of base elements as forming a column vector; and write the multiplication rules (3.4) in the form

$$Au_\theta = P_\theta u_\theta + Q_\theta u_{\theta+1} + R_\theta u_{\theta+2} + \dots, \tag{3.5}$$

where each P_θ is a square matrix and Q_θ, R_θ,... are in general rectangular matrices.

A linear transformation of X affecting only the u_θ's, say

$$v_\theta = Hu_\theta, \qquad u_\theta = H^{-1}v_\theta, \tag{3.6}$$

transforms (3.5) into

$$Av_\theta = HP_\theta H^{-1}v_\theta + HQ_\theta u_{\theta+1} + HR_\theta u_{\theta+2} + \dots, \tag{3.7}$$

and thus induces a collineatory transformation ($HP_\theta H^{-1}$) of the matrix P_θ. Conversely, any such transformation of P_θ corresponds to a linear transformation of the u_θ's.

Take H to be the matrix which reduces P_θ to its classical canonical form.* Let us suppose that the corresponding transformation (3.6) has been carried out, and that this has been done for each of the α sets of nil base elements. We shall then say that the special train algebra X has been given its *canonical form.*

In what follows, it is not essential that the above reduction shall have been carried out completely. It is sufficient if each P_θ has been reduced to the Jacobian form.† That is to say, the essential thing is that the latent roots of each matrix P_θ shall appear down its main diagonal, with zeros below the diagonal.

Let us return to the original notation A, u^σ ($\sigma = 1,\dots, n$) for the basis of X, without, however, changing the order in which the nil base elements have been placed. The multiplication table of X has now the form

$$A^2 = A + \dots, \qquad Au^\sigma = \lambda_\sigma u^\sigma + \dots, \qquad u^\sigma u^\tau = 0.u^\sigma + 0.u^\tau + \dots, \tag{3.8}$$

where in each equation the terms omitted involve only u's with higher suffixes than those written down. (The λ_σ are the latent roots, n in all, of the α matrices P_0, P_1,..., $P_{\alpha-1}$.) Conversely, it is easy to see that a commutative linear algebra whose multiplication table has this form has the defining properties of a special train algebra.

* (7), Ch. VI. † (7), 64.

4. Characteristic and rank equations of a special train algebra

Consider a commutative special train algebra X in the canonical (or partially reduced) form described above. Let its general element be written as

$$x = \xi A + \sum \alpha_\sigma u^\sigma, \tag{4.1}$$

ξ being its weight. Then, using (3.8),

$$\begin{aligned} xA &= \xi A + \dots, \\ xu^\tau &= \xi \lambda_\tau u^\tau + \dots, \end{aligned} \tag{4.2}$$

so that

$$\left.\begin{aligned} 0 &= (\xi - x)A + \dots \\ 0 &= \qquad (\xi\lambda_1 - x)u^1 + \dots \\ 0 &= \qquad\qquad (\xi\lambda_2 - x)u^2 + \dots \quad \text{etc.} \end{aligned}\right\} \tag{4.3}$$

The characteristic equation* of the algebra is obtained by equating to zero x times the determinant of this set of equations; it is therefore

$$x(x-\xi)(x-\lambda_1 \xi)\dots(x-\lambda_n \xi) = 0. \tag{4.4}$$

Powers of x in the expanded form of this equation are to be interpreted as principal powers.†

If the rank equation of X is $f(x) = 0$, then it is known‡ that $f(x)$ must be a factor of the left side of the characteristic equation (4.4). The rank equation therefore has the form

$$x(x-\xi)(x-\mu_1 \xi)\dots(x-\mu_r \xi) = 0, \tag{4.5}$$

where $\mu_1,\dots, \mu_r$ are included in $\lambda_1,\dots, \lambda_n$. X has thus the essential property of a train algebra, and the theorem has been proved.

When multiplication is non-commutative, the third defining postulate of a special train algebra means both

$$\mathsf{X}\mathsf{U}^{(\theta)} \leqslant \mathsf{U}^{(\theta)} \quad (\theta = 1,\dots, \alpha-1) \tag{4.6}$$

and

$$\mathsf{U}^{(\theta)}\mathsf{X} \leqslant \mathsf{U}^{(\theta)} \quad (\theta = 1,\dots, \alpha-1). \tag{4.7}$$

If only (4.6) is assumed, the algebra may be called a *left special train algebra*; the analysis as given leads to a left rank equation, and we find that X is a left train algebra. Similarly, assuming only (4.7), the algebra is a *right special train algebra*; and, using products of the type uA instead of Au, we can show that X is a right train algebra. When both (4.6) and (4.7) hold, the two canonical forms will not necessarily coincide; and in fact the left and right ranks need not be equal.

* (2), § 15, Theorem 3. † (4), § 2. ‡ (1), § 3, Cor. II.

5. An example from genetics*

In (4) § 14 I considered the genetic algebra for the gametic types depending on three linked series of multiple allelomorphs, having the multiplication table (4) (14.3). The transformation which was applied to this had the effect, it will now be seen, of giving this commutative special train algebra its canonical form. Writing u, v, w, p, q, r, s for $\bar{u}, \bar{v}, \bar{w}, \overline{vw}, \overline{wu}, \overline{uv}, \overline{uvw}$, and θ, ϕ, ψ for $1-\omega_{BC}, 1-\omega_{AC}, 1-\omega_{AB}$, the transformed multiplication table is

	I	u	v	w	p	q	r	s
I	I	$\frac{1}{2}u$	$\frac{1}{2}v$	$\frac{1}{2}w$	$\frac{1}{2}\theta p$	$\frac{1}{2}\phi q$	$\frac{1}{2}\psi r$	$\frac{1}{2}\lambda s$
u		0	$\frac{1}{2}\omega_{AB} r$	$\frac{1}{2}\omega_{AC} q$	$\frac{1}{2}\mu s$	0	0	0
v			0	$\frac{1}{2}\omega_{BC} p$	0	$\frac{1}{2}\nu s$	0	0
w				0	0	0	$\frac{1}{2}\rho s$	0
p, q, r, s					0	0	0	0

In the notation of § 3, u, v, w form the u_0 set of base elements; p, q, r are u_1-elements; and s is u_2. The characteristic equation (4.4) is seen to be

$$x(x-\xi)(x-\tfrac{1}{2}\xi)^3(x-\tfrac{1}{2}\theta\xi)(x-\tfrac{1}{2}\phi\xi)(x-\tfrac{1}{2}\psi\xi)(x-\tfrac{1}{2}\lambda\xi) = 0. \qquad (5.1)$$

The rank equation is found as follows. Consider the general element

$$x = \xi I+\alpha u+\beta v+\gamma w+\text{terms in } p, q, r, s,$$

whose square is

$$x^2 = \xi^2 I+\xi\alpha u+\xi\beta v+\xi\gamma w+\text{terms in } p, q, r, s.$$

We have

$$x(x-\xi) = x^2-\xi x = \kappa p+\text{terms in } q, r, s \text{ (say)},$$

$$x(x-\xi)x = \xi\kappa\,.\,\tfrac{1}{2}\theta p+\text{terms in } q, r, s.$$

Hence
$$x(x-\xi)(x-\tfrac{1}{2}\theta\xi) = \text{terms in } q, r, s.$$

Similarly,
$$x(x-\xi)(x-\tfrac{1}{2}\theta\xi)(x-\tfrac{1}{2}\phi\xi) = \text{terms in } r, s,$$

$$x(x-\xi)(x-\tfrac{1}{2}\theta\xi)(x-\tfrac{1}{2}\phi\xi)(x-\tfrac{1}{2}\psi\xi) = \text{a multiple of } s,$$

$$x(x-\xi)(x-\tfrac{1}{2}\theta\xi)(x-\tfrac{1}{2}\phi\xi)(x-\tfrac{1}{2}\psi\xi)(x-\tfrac{1}{2}\lambda\xi) = 0. \qquad (5.2)$$

By considering particular elements such as $I+u$, $I+p$, $I+q$, $I+r$, $I+s$, we can show that no factor in this equation is superfluous, and

* The statement of (4), 247, that all the fundamental symmetrical genetic algebras are special train algebras, refers to the gametic algebras, not to the zygotic algebras which are derived from them by duplication. The latter are train algebras but not in all cases special train algebras.

it is thus the rank equation of the algebra. The principal train roots are therefore 1, $\frac{1}{2}\theta$, $\frac{1}{2}\phi$, $\frac{1}{2}\psi$, $\frac{1}{2}\lambda$.

The plenary rank equation,* or equation of lowest degree connecting plenary powers ($x^{[m]} = x^{2^{m-1}}$) can be obtained most simply by use of *annulling polynomials*, as follows. Consider for simplicity only a normalized element

$$X = I+\alpha u+\beta v+\gamma w+\delta p+\epsilon q+\zeta r+\eta s. \tag{5.3}$$

Let ΦX denote X^2, so that

$$\begin{aligned}\Phi X = I+\alpha u+\beta v+\gamma w+(\delta\theta+\beta\gamma\omega_{BC})p+(...)q+(...)r+\\ +(\eta\lambda+\alpha\delta\mu+\beta\epsilon\nu+\gamma\zeta\rho)s.\end{aligned} \tag{5.4}$$

The operator Φ is to be considered as acting only on the coefficients of X. Its effect on any one coefficient, of course, depends on the values of the other coefficients as well. Thus

$$\left.\begin{aligned}\Phi\alpha = \alpha, \qquad \Phi\beta = \beta, \qquad \Phi\gamma = \gamma\\ \Phi\delta = \delta\theta+\beta\gamma\omega_{BC}, ..., ...\\ \Phi\eta = \eta\lambda+\alpha\delta\mu+\beta\epsilon\nu+\gamma\zeta\rho\end{aligned}\right\}. \tag{5.5}$$

Therefore $(\Phi-1)\alpha = 0$; or, as we may say, the polynomial

$$\Phi-1 \tag{5.6}$$

annuls α; it also annuls β and γ. Also $(\Phi-\theta)\delta = \beta\gamma\omega_{BC}$, which is annulled by $\Phi-1$. Thus, of the polynomials

$$(\Phi-1)(\Phi-\theta), \qquad (\Phi-1)(\Phi-\phi), \qquad (\Phi-1)(\Phi-\psi), \tag{5.7}$$

the first annuls δ, and similarly the other two annul respectively ϵ and ζ. Also

$$(\Phi-\lambda)\eta = \alpha\delta\mu+\beta\epsilon\nu+\gamma\zeta\rho.$$

Each term on the right is annulled by one of the operators (5.7); hence $(\Phi-\lambda)\eta$ is annulled by the L.C.M. of these operators, and η by

$$(\Phi-1)(\Phi-\theta)(\Phi-\phi)(\Phi-\psi)(\Phi-\lambda). \tag{5.8}$$

Finally, the annulling polynomial for X itself is the L.C.M. of the five operators (5.6), (5.7), (5.8). Thus we have the plenary train equation

$$(\Phi-1)(\Phi-\theta)(\Phi-\phi)(\Phi-\psi)(\Phi-\lambda)X = 0, \tag{5.9}$$

i.e.

$$X[X-1][X-\theta][X-\phi][X-\psi][X-\lambda] = 0, \tag{5.10}$$

in which, after expansion, symbolic powers of X are to be interpreted as plenary powers.

We have shown, then, that the sequence of plenary powers forms a train, and that the plenary train roots are 1, θ, ϕ, ψ, λ.

The genetic interpretation of train roots has already been given.†

* See (4), 246–7.

† (4), 247.

Added 16 December 1940. It has been tacitly assumed in § 5 that θ, ϕ, ψ, 1 are all unequal. Genetically, we may suppose without loss of generality that the loci A, B, C are distinct and occur in that order on a chromosome, with

$$0 < \omega_{BC} \leqslant \omega_{AB} < \omega_{AC} < \tfrac{1}{2}.$$

Then it may be shown that

$$\tfrac{1}{4} < \lambda < \phi < \psi \leqslant \theta < 1.$$

Thus an exception to the tacit assumption occurs when $\omega_{BC} = \omega_{AB}$; in this case $\psi = \theta$, the equations (5.2), (5.10) contain repeated factors, and the repetitions are superfluous.

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A NOTE ON THE POLYNOMIALS OF HERMITE

By J. L. BURCHNALL (*Durham*)

[Received 23 October 1940]

FELDHEIM and Watson have recently* given proofs of the formula

$$H_m(x)H_n(x) = m!\,n! \sum_{r=0}^{\min(m,n)} \frac{2^r H_{m+n-2r}(x)}{(m-r)!\,(n-r)!\,r!}, \tag{1}$$

where $H_n(x)$ is Hermite's polynomial, defined by

$$H_n(x) = e^{x^2}(-D)^n e^{-x^2} = (-)^n(D-2x)^n\,1, \qquad D \equiv d/dx. \tag{2}$$

Since Feldheim employs the orthogonal properties of the polynomials, while Watson, starting from the generating function, changes the order of summation in a multiple series, it may be of interest to give a proof depending directly on the definition (2).

If y is any sufficiently differentiable function of x, we have

$$\begin{aligned}(D-2x)^n y &= e^{x^2}D^n e^{-x^2}y \\ &= e^{x^2}\sum_{r=0}^{n}\binom{n}{r}(D^r e^{-x^2})D^{n-r}y \\ &= \sum_{r=0}^{n}(-)^r\binom{n}{r}H_r(x)D^{n-r}y,\end{aligned}$$

i.e.
$$(-)^n(D-2x)^n y = \sum_{r=0}^{n}(-)^r\binom{n}{r}H_{n-r}(x)D^r y. \tag{3}$$

Evidently, from (2),

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x)$$

and, if we observe that

$$(D-2x)^{n+1}\,1 = (D-2x)^n(-2x),$$

and set $y = 2x$ in (3), we have

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

and so
$$H_n'(x) = 2nH_{n-1}(x). \tag{4}$$

These formulae are, of course, well known.

* *J. of London Math. Soc.* 13 (1938), 22–9 and 29–32 respectively.

Now in (3) set $y = H_m(x)$, with $n \leqslant m$, when we obtain

$$\begin{aligned} H_{m+n}(x) &= (-)^n(D-2x)^n H_m(x) \\ &= \sum_{r=0}^{n} (-)^r \binom{n}{r} H_{n-r}(x) D^r H_m(x) \\ &= \sum_{r=0}^{n} (-)^r \binom{n}{r} H_{n-r}(x) \frac{m!}{(m-r)!} 2^r H_{m-r}(x), \quad \text{by (4).} \end{aligned}$$

Thus

$$H_{m+n}(x) = m!\,n! \sum_{r=0}^{\min(m,n)} \frac{(-)^r 2^r}{(m-r)!\,(n-r)!\,r!} H_{m-r}(x) H_{n-r}(x). \tag{5}$$

The formulae (1) and (5) may be regarded as inverse to one another and (1) may now be derived either by inverting (5) or by employing this latter formula on the right-hand side of (1), thus obtaining a verification. It is, however, possible to prove (1) directly by employing the theory of adjoint operators.*

If we regard Leibniz's theorem

$$D^n(vw) = \sum_{r=0}^{n} \binom{n}{r} (D^r v) D^{n-r} w$$

as expressing the equivalence of certain operations on the function w, then, taking the adjoints of these operations, we have

$$vD^n w = \sum_{r=0}^{n} (-)^r \binom{n}{r} D^{n-r}(wD^r v).$$

Setting $e^{-x^2}w$ in place of w we get

$$v(D-2x)^n w = \sum_{r=0}^{n} (-)^r \binom{n}{r} (D-2x)^{n-r}(wD^r v).$$

In this set $v = H_m(x)$, $w = 1$ $(m \geqslant n)$, and use (2) and (4). Then

$$\begin{aligned} (-)^n H_m(x) H_n(x) &= \sum_{r=0}^{n} (-)^r \binom{n}{r} \frac{2^r m!}{(m-r)!} (D-2x)^{n-r} H_{m-r}(x) \\ &= (-)^n \sum_{r=0}^{n} \binom{n}{r} \frac{2^r m!}{(m-r)!} H_{m+n-2r}(x), \end{aligned}$$

which is (1).

* I am indebted to Mr. T. W. Chaundy for suggesting this method of attack.

Similar results hold for the polynomials

$$\mathfrak{H}_n(x) = (-)^n e^{\frac{1}{2}x^2} D^n e^{-\frac{1}{2}x^2} = (-)^n (D-x)^n \, 1,$$

the formulae corresponding to (3), (5), (1) being

$$(-)^n (D-x)^n y = \sum_{r=0}^{n} (-)^r \binom{n}{r} \mathfrak{H}_{n-r}(x) D^r y, \tag{6}$$

$$\mathfrak{H}_{m+n}(x) = m!\, n! \sum_{r=0}^{\min(m,n)} \frac{(-)^r \mathfrak{H}_{m-r}(x) \mathfrak{H}_{n-r}(x)}{(m-r)!\,(n-r)!\,r!}, \tag{7}$$

$$\mathfrak{H}_m(x) \mathfrak{H}_n(x) = m!\, n! \sum_{r=0}^{\min(m,n)} \frac{\mathfrak{H}_{m+n-2r}(x)}{(m-r)!\,(n-r)!\,r!}. \tag{8}$$

We may further note that, if in (6) we interchange r, $n-r$ and set $e^{-\frac{1}{2}x^2}y$ for y, we obtain

$$(D-2x)^n y = \sum_{r=0}^{n} (-)^r \binom{n}{r} \mathfrak{H}_r(x) (D-x)^{n-r} y,$$

and, taking $y = 1$, we have the known formula

$$H_n(x) = \sum_{r=0}^{n} \binom{n}{r} \mathfrak{H}_r(x) \mathfrak{H}_{n-r}(x). \tag{9}$$

Finally, we may note the following identities between operators immediately deducible from (3) and (6),

$$D^n y = \sum_{r=0}^{n} (-)^r \binom{n}{r} H_r(x) (D+2x)^{n-r} y, \tag{10}$$

$$D^n y = \sum_{r=0}^{n} (-)^r \binom{n}{r} \mathfrak{H}_r(x) (D+x)^{n-r} y, \tag{11}$$

with the adjoint identities

$$D^n y = \sum_{r=0}^{n} \binom{n}{r} (D-2x)^{n-r} H_r(x) y, \tag{12}$$

$$D^n y = \sum_{r=0}^{n} \binom{n}{r} (D-x)^{n-r} \mathfrak{H}_r(x) y. \tag{13}$$

ON THE DOUBLE-INTEGRAL REPRESENTATION OF APPELL'S FUNCTION F_4

By W. N. BAILEY (*Manchester*)

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1. SOME years ago I gave the formula*

$$F_4[a,b;c,a+b-c+1;x(1-y),y(1-x)] = F(a,b;c;x)F(a,b;a+b-c+1;y) \quad (1.1)$$

valid inside simply-connected regions surrounding $x = 0$, $y = 0$ for which

$$|x(1-y)|^{\frac{1}{2}}+|y(1-x)|^{\frac{1}{2}} < 1.$$

Recently Burchnall and Chaundy† have proved the more general result

$$F_4[a,b;c,c';x(1-y),y(1-x)] = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r} \times$$
$$\times x^r y^r F(a+r,b+r;c+r;x)F(a+r,b+r;c'+r;y), \quad (1.2)$$

which reduces to (1.1) when $c+c' = a+b+1$.

From (1.2) the authors deduced, by substituting integrals for the hypergeometric functions on the right, and summing, that

$$F_4[a,b;c,c';x(1-y),y(1-x)] = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)} \times$$
$$\times \int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1} \times$$
$$\times (1-ux)^{a-c-c'+1}(1-vy)^{b-c-c'+1}(1-ux-vy)^{c+c'-a-b-1}\,dudv, \quad (1.3)$$

provided that $R(c) > R(a) > 0$, $R(c') > R(b) > 0$, and $|x| \leqslant \rho$, $|y| \leqslant \rho'$ where $\{\rho(1+\rho')\}^{\frac{1}{2}}+\{\rho'(1+\rho)\}^{\frac{1}{2}} < 1$.

The formula (1.3) appears to be the first double-integral representation of this type to be given for Appell's function F_4, although similar, but simpler, representations have been known for many years in the cases of the other three types of Appell's functions.

The proof of (1.2) given by Burchnall and Chaundy depends on transformations of hypergeometric series. It may therefore be of interest to give a proof which depends only on well-known formulae

* See § 9.6 of my tract *Generalized Hypergeometric Series* (Cambridge, 1935).

† J. L. Burchnall and T. W. Chaundy, *Quart. J. of Math.* (Oxford), 11 (1940), 249–70.

for ordinary hypergeometric series, together with the simple theorem of Saalschütz for the sum of a terminating ${}_3F_2$.

2. We consider the function

$$(1-x)^{-a}(1-y)^{-b}F_4\left[a,b;c,c';-\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right],$$

which is an analytic function of x and y when $|x|$ and $|y|$ are sufficiently small, and can therefore be expanded in a double series of powers of x and y. The coefficient of $x^m y^n$ in this expansion is*

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(a)_{r+s}(b)_{r+s}}{r!\,s!\,(c)_r(c')_s}\,\frac{(-1)^{r+s}(a+r+s)_{m-r}(b+r+s)_{n-s}}{(m-r)!\,(n-s)!}$$

$$=\frac{(a)_m(b)_n}{m!\,n!}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(a+m)_s(b+n)_r(-m)_r(-n)_s}{r!\,s!\,(c)_r(c')_s}$$

$$=\frac{(a)_m(b)_n}{m!\,n!}F(a+m,-n;c')F(b+n,-m;c)$$

$$=\frac{(a)_m(b)_n}{m!\,n!}\,\frac{(c'-a-m)_n}{(c')_n}\,\frac{(c-b-n)_m}{(c)_m}.$$

Now by Saalschütz's theorem

$$\frac{(c'-a-m)_n(c-b-n)_m}{(c'-a)_n(c-b)_m}={}_3F_2\left[\begin{matrix}a+b-c-c'+1, & -m, -n;\\ 1-c+b-m, & 1-c'+a-n\end{matrix}\right],$$

and so

$$(1-x)^{-a}(1-y)^{-b}F_4\left[a,b;c,c';-\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n(c-b)_m(c'-a)_n}{m!\,n!\,(c)_m(c')_n}x^m y^n\times$$

$$\times\sum_{r=0}^{\min(m,n)}\frac{(a+b-c-c'+1)_r(-m)_r(-n)_r}{r!(1-c+b-m)_r(1-c'+a-n)_r}.$$

Putting $m=r+s$, $n=r+t$, and interchanging the order of summation, we get

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(a)_{r+s}(b)_{r+t}(c-b)_s(c'-a)_t(a+b-c-c'+1)_r}{r!\,s!\,t!\,(c)_{r+s}(c')_{r+t}}x^{r+s}y^{r+t}$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r}x^r y^r F(a+r,c-b;c+r;x)\times$$

$$\times F(b+r,c'-a;c'+r;y),$$

* Cf. the proof of (1.1) given in my tract.

and using the well-known formula

$$F(a,b;c;x) = (1-x)^{-a}F\left(a, c-b; c; -\frac{x}{1-x}\right)$$

we obtain (1.2) with $-x/(1-x)$, $-y/(1-y)$ written in place of x, y.

The proof is easily justified if $|x| < \rho$, $|y| < \rho$, where $\rho = 3-2\sqrt{2}$, in which case

$$|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < 2\rho^{\frac{1}{2}} = 1-\rho < |(1-x)(1-y)|^{\frac{1}{2}}.$$

The complete result then follows by an appeal to the theory of analytic continuation.

3. In the above proof we have obtained the formula

$$(1-x)^{-a}(1-y)^{-b}F_4\left[a,b;c,c';-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right]$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r}x^ry^rF(a+r,c-b;c+r;x)\times$$
$$\times F(b+r,c'-a;c'+r;y). \qquad (3.1)$$

If we replace the hypergeometric functions on the right by integral representations, and sum with respect to r, we find that

$$(1-x)^{-a}(1-y)^{-b}F_4\left[a,b;c,c';-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right]$$
$$=\frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)}\times$$
$$\times\int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{b-c}\times$$
$$\times(1-vy)^{a-c'}(1-uvxy)^{c+c'-a-b-1}\,dudv, \qquad (3.2)$$

provided that $R(c) > R(a) > 0$, $R(c') > R(b) > 0$, and x, y lie inside simply-connected regions surrounding $x = 0$, $y = 0$ for which $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < |(1-x)(1-y)|^{\frac{1}{2}}$. The last condition is satisfied, for example, if $|x| \leqslant \rho$, $|y| \leqslant \rho'$ where $\rho^{\frac{1}{2}}+\rho'^{\frac{1}{2}} < \{(1-\rho)(1-\rho')\}^{\frac{1}{2}}$.

Mr. Chaundy has pointed out to me that (1.3) and (3.2) transform into each other by the substitution

$$u = \frac{u'(1-x')}{1-u'x'}, \qquad v = \frac{v'(1-y')}{1-v'y'},$$

where $$x = -\frac{x'}{1-x'}, \qquad y = -\frac{y'}{1-y'}.$$

THE ABSOLUTE CESÀRO-SUMMABILITY PROBLEM FOR DIFFERENTIATED FOURIER SERIES

By L. S. BOSANQUET (*London*)

[Received 19 December 1940]

1. Introduction. I have shown in a previous paper* that *a necessary and sufficient condition that the series obtained by formally differentiating the Fourier series of a Lebesgue-integrable function $f(t)$ should be summable $(C,\alpha+1)$ at $t = x$, where $\alpha \geqslant 0$, is that the even function*† $\psi(t)/(2\sin\frac{1}{2}t) = \{f(x+t)-f(x-t)\}/(4\sin\frac{1}{2}t)$ *should be integrable in the Cesàro-Lebesgue sense in* $(0,\pi)$ *and its Fourier series be summable* (C,α) *at* $t = 0$ *to the same sum.*

The object of the present paper is to show that the problem of the absolute Cesàro summability‡ of a differentiated Lebesgue-Fourier series may be reduced in an analogous way to the problem of the absolute Cesàro summability of a Cesàro-Lebesgue Fourier series.§

The direct analogue of the (C) theorem is established for $\alpha > 1$ (Theorem 1). The problem of whether Theorem 1 is true when $0 < \alpha \leqslant 1$ is unsettled, but an example suggested by Dr. Hyslop shows that it is false when $\alpha = 0$.|| If, however, the function

* *Quart. J. of Math.* (Oxford), 10 (1939), 67–74. See also *Proc. London Math. Soc.* (2) 46 (1940), 270–89, where the corresponding problem for successively derived Fourier series is solved, and the analogous result for conjugate series stated. These two papers will be referred to as DFS and D_rFS respectively.

† I write $\psi(t) = \frac{1}{2}\{f(x+t)-f(x-t)\}$ throughout the paper. By the Fourier series of a non-periodic function we mean that of the corresponding function defined by periodicity outside $(-\pi,\pi)$.

‡ The series $\sum_{0}^{\infty} u_n$ is said to be *summable* $|C,\alpha|$ to s, where $\alpha > -1$, if $s_n^\alpha \to s$ and $\sum |\Delta s_n^\alpha|$ is convergent, where $s_n^\alpha = \dfrac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^\alpha u_\nu$ is the nth Cesàro mean of order α of the series, and $\Delta v_n = v_n - v_{n+1}$. The sequence $s_n = u_0+...+u_n$ is said to *tend to* s $|C,\alpha|$.

§ Sufficient conditions and necessary conditions for the summability $|C|$ of a derived Fourier series have been given by Hyslop, under the restriction that $\psi(t)/t$ be integrable L. See J. M. Hyslop, *Proc. London Math. Soc.* (2) 46 (1940), 55–80, where corresponding results for successively derived Fourier series and conjugate series are given.

|| Taking $f(t) = \psi(t) = \frac{1}{2}t$ in $(-\pi,\pi)$, we see that the differentiated Fourier series is $\sum(-1)^{n-1}\cos nt$, which is not summable $|C,1|$ at $t = 0$, but the Fourier series of $t/(4\sin\frac{1}{2}t)$ is absolutely convergent. Cf. Hyslop, loc. cit. 79.

$g(t) = \psi(t)/(2\sin\frac{1}{2}t)$ is replaced by* $G(t) = \psi(2t)/(2\sin t)$ the result obtained is valid for $\alpha \geqslant 0$ (Theorem 2). The latter function has an additional point of non-absolute integrability at $t = \pi$, if it has one at $t = 0$. This is irrelevant when $\alpha > 1$, since the problem is then a 'local' one. When $\alpha > 1$ the function $g(t)$ may also be replaced by $\gamma(t) = \psi(t)/t$, since $g(t) - \gamma(t)$ satisfies Dini's convergence criterion, and, *a fortiori*, de la Vallée Poussin's criterion, which is sufficient for the summability $|C, 1+\delta|$ $(\delta > 0)$ of its Fourier series.†

In order to preserve the analogy between the proofs of the (C) theorems and the $|C|$ theorems I introduce a new integral which I call the *absolute Cesàro-Lebesgue integral*. This integral bears the same relation to the Lebesgue integral as the Cesàro-Lebesgue does to the Cauchy-Lebesgue. The relation between the Cesàro-Lebesgue and the absolute Cesàro-Lebesgue integrals may be expressed by saying that the former is 'C-continuous', while the latter is '$|C|$-continuous', i.e. its mean value of some order is of bounded variation as well as continuous.

2. The absolute Cesàro-Lebesgue integral

2.1. Let λ denote a non-negative integer.

If $h(t)$ is of bounded variation in an open interval $(0, \eta)$ $(\eta > 0)$ and tends to s as $t \to +0$, we say that *$h(t)$ tends absolutely to s* as $t \to +0$, and write $h(t) \to s\ |C, 0|$.

Suppose that $g(t)$ is integrable L in (ϵ, a), for every ϵ such that $0 < \epsilon < a$ (a fixed). *If there is a continuous function $G_{\lambda+1}(t)$ such that*

(i) $(d/dt)^{\lambda+1}G_{\lambda+1}(t) = g(t)$ *for almost all t in* $(0, a)$,

(ii) $t^{-\lambda}G_{\lambda+1}(t) \to 0\ |C, 0|$ *as* $t \to +0$,

we say that $g(t)$ is integrable $|C_\lambda L|$ in $(0, a)$, i.e. in the absolute Cesàro-Lebesgue sense of order λ.

If $g(t)$ is integrable $|C_\lambda L|$ in $(0, a)$ it is also integrable $C_\lambda L$, and *the value of the $|C_\lambda L|$ integral is defined as that of the $C_\lambda L$ integral.*‡ For completeness we give below a definition of the value which does not mention the $C_\lambda L$ integral.

If $g(t)$ is integrable $|C_\lambda L|$ for an *unspecified* λ we say that it is *integrable* $|CL|$.

* This function was employed by W. H. Young in investigating derived Fourier series, *Proc. London Math. Soc.* (2) 17 (1918), 195–236.

† L. S. Bosanquet, *J. of London Math. Soc.* 11 (1936), 11–15.

‡ See DFS or D_rFS.

2.2. *If $g(t)$ is integrable $|C_\lambda L|$ in $(0,a)$, then it is also integrable $|C_{\lambda+1} L|$ in $(0,a)$.*

For let
$$G_{\lambda+2}(t) = \int_0^t G_{\lambda+1}(u)\,du.$$
Then plainly $G_{\lambda+2}(t)$ is continuous and $(d/dt)^{\lambda+2}G_{\lambda+2}(t) = g(t)$ for almost all t in $(0,a)$, while $t^{-\lambda-1}G_{\lambda+2}(t) = o(1)$ as $t \to +0$. Now it is easily verified that*
$$\frac{d}{dt}\{t^{-\lambda-1}G_{\lambda+2}(t)\} = t^{-\lambda-2}\int_0^t u^{\lambda+1}\,d\{u^{-\lambda}G_{\lambda+1}(u)\}$$
for all t in the open interval $(0,a)$, and hence, if $0 < \eta < a$,
$$\begin{aligned}\lim_{\epsilon\to+0}\int_\epsilon^\eta |d\{t^{-\lambda-1}G_{\lambda+2}(t)\}| &= \int_0^\eta t^{-\lambda-2}\,dt\left|\int_0^t u^{\lambda+1}\,d\{u^{-\lambda}G_{\lambda+1}(u)\}\right| \\ &\leqslant \int_0^\eta t^{-\lambda-2}\,dt\int_0^t u^{\lambda+1}\,|d\{u^{-\lambda}G_{\lambda+1}(u)\}| \\ &= \int_0^\eta |d\{u^{-\lambda}G_{\lambda+1}(u)\}|\,u^{\lambda+1}\int_u^\eta t^{-\lambda-2}\,dt \\ &\leqslant \frac{1}{\lambda+1}\int_0^\eta |d\{u^{-\lambda}G_{\lambda+1}(u)\}|,\end{aligned}$$
whence the result follows.†

If $g(t)$ is integrable $|C_\lambda L|$ in $(0,a)$, we write
$$G_s(t) = (d/dt)^{\lambda+1-s}G_{\lambda+1}(t)$$
for $s = 1, 2,\ldots, \lambda$, where $0 < t < a$. Thus $G_s(t)$ *is integrable* $|C_{\lambda-s} L|$ in $(0,a)$. The function $G_s(t)$ is unchanged if λ is replaced by $\lambda+1$, and is thus uniquely defined for all positive integral values of s.

If $g(t)$ is integrable $|CL|$ in $(0,a)$, $p > -1$, and
$$\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}t^{-\lambda-p}G_\lambda(t) \to s\ |C,0|$$
as $t \to +0$, we write $g(t) \sim st^p|C,\lambda|$ as $t \to +0$. The related notation, such as $g(t) = o(t^p)\ |C,\lambda|$, is interpreted in the obvious way.

If $g(t) \sim st^p\ |C,\lambda|$ as $t \to +0$, the same is true if λ is replaced by $\lambda+1$.†

* For convenience of notation we define $u^{-\lambda}G_{\lambda+1}(u)$ as zero at $u = 0$.

† Cf. Hyslop, loc. cit., Lemma 12.

If $g(t)$ is integrable $|CL|$ in $(0,a)$, and $g(t) \sim st^p\,|C,\lambda+1|$ as $t \to +0$, where $p > -1$, then $g(t)$ is necessarily integrable $|C_\lambda L|$.

If $g(t)$ is integrable $|C_\lambda L|$, then $G_s(t) = o(t^{s-1})\ |C,\lambda-s+1|$ as $t \to +0$, for $s = 1, 2,\ldots, \lambda$. In particular $G_1(t) = o(1)\ |C,\lambda|$.

The proofs of these statements are left to the reader.

We may now define *the value of the $|C_\lambda L|$ integral of $g(t)$ over $(0,a)$ by the equation*

$$\int_{\to 0|C,\lambda|}^{a} g(t)\,dt = |C,\lambda|\text{-}\lim_{\epsilon\to+0}\int_{\epsilon}^{a} g(t)\,dt.$$

The $|C_0 L|$ integral is equivalent to the L integral.

3. Certain lemmas

In this section we give as lemmas some results which we shall require later.

LEMMA 1. *If $p > -1$, $r < p+1$, $g(t)$ is integrable $|C_\lambda L|$ in $(0,a)$ and $g(t) = o(t^p)\ |C,\lambda+1|$ as $t \to +0$, and if $t^{r+\rho}h^{(\rho)}(t)$ is of bounded variation in $(0,a)$ for $\rho = 0, 1,\ldots, \lambda+1$, then $g(t)h(t)$ is integrable $|C_\lambda L|$ in $(0,a)$ and is $o(t^{p-r})\ |C,\lambda+1|$ as $t \to +0$.*

If $0 < \epsilon \leqslant t \leqslant a$, we have

$$\int_{\epsilon}^{t} g(u)h(u)\,du = \big[G_1(u)h(u)\big]_{\epsilon}^{t} - \int_{\epsilon}^{t} G_1(u)h'(u)\,du. \tag{3.1}$$

Suppose $\lambda = 0$. Then $G_1(t) = o(t^{p+1})\ |C,0|$, and hence, using the fact that the product of two functions of bounded variation is itself of bounded variation, we see that $t^{-p-1}G_1(t).t^r h(t) = o(1)\ |C,0|$ as $t \to +0$, i.e. $G_1(t)h(t) = o(t^{p+1-r})\ |C,0|$.

Similarly, $G_1(t)h'(t) = o(t^{p-r})\ |C,0|$, and hence $G_1(t)h'(t) = o(t^{p-r})$ $|C,1|$ as $t \to +0$, so that $G_1(t)h'(t)$ is integrable $|C_0 L|$, i.e. integrable L, and

$$\int_{0}^{t} G_1(u)h'(u)\,du = o(t^{p+1-r})\ |C,0|.$$

It follows that $g(t)h(t)$ is integrable $|C_0 L|$ in $(0,a)$, and

$$\int_{0}^{t} g(u)h(u)\,du = o(t^{p+1-r})\ |C,0|,$$

i.e. $g(t)h(t) = o(t^{p-r})\ |C,1|$ as $t \to +0$. Thus the lemma is proved when $\lambda = 0$.

The result in the general case is obtained from (3.1) by induction. The argument is similar to that used for the analogous result with (C) in place of $|C|$, and the details are therefore omitted.*

LEMMA 2. *If $g(t)$ is integrable $|C_\lambda L|$ in $(0,a)$, then so are $g(t)\cos\nu t$ and $g(t)\sin\nu t$, where ν is real, and*

$$\int_{\to 0|C,\lambda|}^{a} g(t)e^{i\nu t}\,dt = G_1(a)e^{i\nu a}-i\nu \int_{\to 0|C,\mu|}^{a} G_1(t)e^{i\nu t}\,dt,$$

where $\mu = \max(\lambda-1, 0)$.

The proof is similar to that of the (C) analogue.†

LEMMA 3. *If $\phi(t)$ is integrable L in $(0,\pi)$, and*

$$\alpha_n+i\beta_n = \int_0^\pi \phi(t)e^{int}\,dt,$$

then α_n and β_n are $o(1)\ |C, 1+\delta|$ as $n\to\infty$, for $\delta > 0$.

We may suppose without loss of generality that $0<\delta<1$. Writing α_n^σ, $c^\sigma(n,t)$ for the nth Cesàro means of order σ of the sequences α_n and $\cos nt$ respectively, we have

$$\alpha_n^{1+\delta} = \int_0^\pi \phi(t)c^{1+\delta}(n,t)\,dt,$$

and hence

$$\begin{aligned}\sum_{n=0}^{\infty}|\Delta\alpha_n^{1+\delta}| &= \sum_{n=0}^{\infty}\left|\int_0^\pi \phi(t)\Delta c^{1+\delta}(n,t)\,dt\right|\\ &\leqslant \sum_{n=0}^{\infty}\int_0^\pi |\phi(t)||\Delta c^{1+\delta}(n,t)|\,dt\\ &= \int_0^\pi |\phi(t)|\left\{\sum_{n=0}^{\infty}|\Delta c^{1+\delta}(n,t)|\right\}dt\\ &\leqslant A\int_0^\pi |\phi(t)|\,dt < \infty,\end{aligned}$$

* Cf. DFS, Lemma 1. We simply replace (C) by $|C|$ in the proof of the (C) analogue.

† Cf. DFS, Lemma 2.

since* $$\sum |\Delta c^{1+\delta}(n,t)| = \sum_{n\leqslant t^{-1}} O(nt^2)+\sum_{n>t^{-1}} O(n^{-1-\delta}t^{-\delta})$$
$$= O(1)+O(1)$$
for $0 < t < \pi$.

Since $\alpha_n = o(1)$ by the Riemann-Lebesgue theorem, it follows that $\alpha_n = o(1)\ |C,1+\delta|$ as $n \to \infty$.

Similarly, it may be shown that $\beta_n = o(1)\ |C,1+\delta|$.†

4. The main theorems

THEOREM 1. *If $\alpha > 1$, and if $f(t)$ is integrable L in $(-\pi,\pi)$ and of period 2π, then a necessary and sufficient condition that the differentiated Fourier series of $f(t)$ should be summable $|C,\alpha+1|$ to sum s at the point $t = x$ is that the even function $\{f(x+t)-f(x-t)\}/(4\sin\frac{1}{2}t)$ should be integrable $|CL|$ in $(0,\pi)$ and its Fourier series‡ summable $|C,\alpha|$ to sum s at $t = 0$.*

* We have $$\Delta c^{1+\delta}(n,t) = \frac{2\sin\frac{1}{2}t}{n+1} d_{n+1}^{1+\delta},$$
where $d_n = n\sin(n-\frac{1}{2})t = O(n^2t)$, and hence $\Delta c^{1+\delta}(n,t) = O(nt^2)$.
On the other hand,
$$\Delta c^{1+\delta}(n,t) = (1+\delta)\frac{c^{1+\delta}(n+1,t)-c^{\delta}(n+1,t)}{n+1},$$
and $$c^{\sigma}(n,t) = \frac{\sigma}{n+\sigma}\{\tfrac{1}{2}+\kappa^{\sigma-1}(n,t)\},$$
where $\kappa^{\sigma-1}(n,t)$ is Fejér's kernel of order $\sigma-1$, which is $O(n^{1-\sigma}t^{-\sigma})$, if $0<\sigma<2$, and hence, for $nt \geqslant 1$, $\Delta c^{1+\delta}(n,t) = O(n^{-1-\delta}t^{-\delta})$. Cf. S. P. Bhatnagar, *Proc. Edinburgh Math. Soc.* (in the press).

† The stronger result that $\sum n^{-1}\beta_n$ is summable $|C,\delta|$ $(\delta > 0)$ is also true, since $\sum n^{-1}\beta_n \cos nt = A-\pi\int_0^{|t|}\phi(u)\,du$, which is of bounded variation in $(0,\pi)$. Cf. L. S. Bosanquet, *J. of London Math. Soc.*, loc. cit., Theorem 1, and L. S. Bosanquet and J. M. Hyslop, *Math. Zeits.* 42 (1937), 489–512, Lemma 21.
On the other hand, Lemma 3 is false with $\delta = 0$. In fact
$$\sum |\Delta\alpha_n^1| = \sum\left|\int_0^{\pi}\phi(t)\,\Delta\left(\frac{1}{n+1}\left\{\frac{1}{2}+\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right\}\right)\right| = \infty$$
for some $\phi(t)$ integrable L in $(0,\pi)$, since $\sum\left|\Delta\left(\frac{1}{n+1}\left\{\frac{1}{2}+\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right\}\right)\right| = \infty$ for $0<t<\pi$. Cf. L. S. Bosanquet and H. Kestelman, *Proc. London Math. Soc.* (2) 45 (1939), 88–97.

‡ i.e. $\frac{1}{2}\alpha(0)+\sum_{n=1}^{\infty}\alpha(n)\cos nt$, where $\alpha(n) = \frac{2}{\pi}\int_{\to 0|C|}^{\pi}\frac{\psi(t)}{2\sin\frac{1}{2}t}\cos nt\,dt$.

We write $$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Then the differentiated Fourier series of $f(t)$ is

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t),$$

and $$\psi(t) \sim \sum_{n=1}^{\infty} B_n \sin nt,$$

where $B_n = B_n(x)$.

Lemma 4. *If $f(t)$ is integrable L in $(-\pi, \pi)$ and $\sum nB_n$ is summable $|C|$ to sum s, then $\psi(t) \sim st\ |C|$ as $t \to +0$.*

Suppose that $\sum nB_n$ is summable $|C, \kappa|$ to s, where κ is a positive integer, and write S_n^σ for the nth Cesàro sum of order σ of $\sum nB_n$, and $s_n^\sigma = S_n^\sigma / A_n^\sigma$ for the corresponding Cesàro mean. Then, for $t > 0$,*

$$\begin{aligned}(\kappa+1)!\frac{\Psi_{\kappa+1}(t)}{t^{\kappa+2}} &= \frac{\kappa+1}{t}\int_0^1 (1-u)^\kappa \psi(tu)\,du \\ &= \frac{\kappa+1}{t}\sum_{n=1}^{\infty} B_n \bar{\gamma}_{\kappa+1}(nt) \\ &= \sum_{n=1}^{\infty} nB_n \gamma_{\kappa+2}(nt) \\ &= \sum_{n=1}^{\infty} S_n^\kappa \Delta^{\kappa+1}\gamma_{\kappa+2}(nt) \\ &= \sum_{n=1}^{\infty} s_n^\kappa A_n^\kappa \Delta^{\kappa+1}\gamma_{\kappa+2}(nt) \\ &= -\sum_{n=1}^{\infty} \Delta s_{n-1}^\kappa J(n,t),\end{aligned}$$

where $$J(n,t) = \sum_{\nu=n}^{\infty} A_\nu^\kappa \Delta^{\kappa+1}\gamma_{\kappa+2}(\nu t). \tag{4.1}$$

If we now show that

$$\left|\frac{\partial}{\partial t}J(n,t)\right| \leqslant \begin{cases} An^{\kappa+2}t^{\kappa+1}, \\ An^{-1}t^{-2} \end{cases} \tag{4.2}$$

* Cf. DFS, Lemma 3. We write, for $\sigma > 0$,

$$\gamma_\sigma(x) + i\bar{\gamma}_\sigma(x) = \int_0^1 (1-u)^{\sigma-1}e^{ixu}\,du.$$

Then $$\Delta^\kappa(\partial/\partial t)^\rho \gamma_\sigma(nt) = O(t^\kappa n^\rho)\min[O(1), O\{(nt)^{-\kappa-\rho-2} + (nt)^{-\sigma}\}].$$

Cf. L. S. Bosanquet, *Proc. London Math. Soc.* (2) 31 (1930), 144–64. The partial summations are justified since $B_n = o(1)$ and $s_n^\kappa = O(1)$.

for $0 < t \leqslant \pi$, $n = 1, 2, \ldots$, it will follow* that, for $0 < t \leqslant \pi$,

$$(\kappa+1)!\frac{d}{dt}\left\{\frac{\Psi_{\kappa+1}(t)}{t^{\kappa+2}}\right\} = -\sum_{n=1}^{\infty} \Delta s_{n-1}^{\kappa}\frac{\partial}{\partial t}J(n,t),$$

and we shall have

$$(\kappa+1)!\lim_{\epsilon\to+0}\int_{\epsilon}^{\pi}\left|d\left\{\frac{\Psi_{\kappa+1}(t)}{t^{\kappa+2}}\right\}\right| = \int_0^{\pi}\left|\sum_{n=1}^{\infty}\Delta s_{n-1}^{\kappa}\frac{\partial}{\partial t}J(n,t)\right|dt$$

$$\leqslant \int_0^{\pi}\sum_{n=1}^{\infty}|\Delta s_{n-1}^{\kappa}|\left|\frac{\partial}{\partial t}J(n,t)\right|dt$$

$$= \sum_{n=1}^{\infty}|\Delta s_{n-1}^{\kappa}|\int_0^{\pi}\left|\frac{\partial}{\partial t}J(n,t)\right|dt$$

$$\leqslant A\sum_{n=1}^{\infty}|\Delta s_{n-1}^{\kappa}| < \infty,$$

since

$$\int_0^{\pi}\left|\frac{\partial}{\partial t}J(n,t)\right|dt = \int_0^{n^{-1}}O(n^{\kappa+2}t^{\kappa+1})\,dt + \int_{n^{-1}}^{\pi}O(n^{-1}t^{-2})\,dt$$

$$= O(1)+O(1)$$

for $n = 1, 2, \ldots$.

Now we have proved in DFS that $\psi(t) \sim st\ (C,\kappa+1)$, and hence it will follow that $\psi(t) \sim st\ |C,\kappa+1|$ as $t \to +0$.

To establish the inequalities (4.2), we first observe that†

$$J(0,t) = \sum_{\nu=0}^{\infty}A_{\nu}^{\kappa}\Delta^{\kappa+1}\gamma_{\kappa+2}(\nu t)$$

$$= \sum_{\nu=0}^{\infty}\Delta\gamma_{\kappa+2}(\nu t) = 1,$$

and hence

$$\frac{\partial}{\partial t}J(n,t) = \frac{\partial}{\partial t}\{J(n,t)-J(0,t)\}$$

$$= -\sum_{\nu=0}^{n-1}A_{\nu}^{\kappa}\Delta^{\kappa+1}\frac{\partial}{\partial t}\gamma_{\kappa+2}(\nu t)$$

$$= \sum_{\nu=0}^{n-1}O(\nu^{\kappa})O(\nu t^{\kappa+1})$$

$$= O(n^{\kappa+2}t^{\kappa+1}).$$

* By uniform convergence for $t \geqslant \epsilon > 0$, since $\sum|\Delta s_n^{\kappa}| < \infty$.

† e.g. by κ partial summations.

On the other hand, writing

$$J(n,t) = A_n^\kappa \Delta^\kappa \gamma_{\kappa+2}(nt) + \sum_{\nu=n+1}^{\infty} A_\nu^{\kappa-1} \Delta^\kappa \gamma_{\kappa+2}(\nu t),$$

we obtain,* for $0 < t \leqslant \pi$, $n = 1, 2,...,$

$$\begin{aligned}\frac{\partial}{\partial t} J(n,t) &= A_n^\kappa \Delta^\kappa \frac{\partial}{\partial t}\gamma_{\kappa+2}(nt) + \sum_{\nu=n+1}^{\infty} A_\nu^{\kappa-1}\Delta^\kappa \frac{\partial}{\partial t}\gamma_{\kappa+2}(\nu t) \\ &= O(n^\kappa)O(n^{-\kappa-1}t^{-2}) + \sum_{\nu=n+1}^{\infty} O(\nu^{\kappa-1})O(\nu^{-\kappa-1}t^{-2}) \\ &= O(n^{-1}t^{-2}),\end{aligned}$$

which completes the proof of the lemma.

Lemma 5. *If $\psi(t)$ is integrable L in $(0,\pi)$, and if $\psi(t) \sim st$ $|C|$ as $t \to +0$, then $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable $|CL|$ in $(0,\pi)$ and tends to s $|C|$* as $t \to +0$.

This is a corollary of Lemma 1.†

Lemma 6. *If $g(t)$ is even and integrable $|CL|$ in $(0,\pi)$, and if $g(t) \to s$ $|C|$ as $t \to +0$, then the Fourier series of $g(t)$ is summable $|C|$ to s at $t = 0$.*

This has been proved elsewhere‡ in the case when $g(t)$ is integrable L. The same proof holds in the general case if Lemma 2 takes the place of integration by parts.

Lemma 7. *If $g(t)$ is integrable $|CL|$ in $(0,\pi)$ and $tg(t)$ is integrable L, and if*

$$\alpha(\mu) = \frac{2}{\pi}\int_{\to 0|C|}^{\pi} g(t)\cos\mu t\,dt,$$

then, if one of the series $\frac{1}{2}\alpha(0) + \sum_{n=1}^{\infty}\alpha(n)$ and $\sum_{n=0}^{\infty}\alpha(n+\frac{1}{2})$ is summable $|C,\beta|$ to sum s, where $\beta > 1$, so is the other.

For, if the sums of the first $n+1$ terms of the series are denoted by σ_n and τ_n respectively, we have§

$$\begin{aligned}2\tau_{n-1} - \sigma_n - \sigma_{n-1} &= \frac{2}{\pi}\int_0^\pi g(t)\tan\tfrac{1}{4}t\sin nt\,dt \\ &= o(1)\ |C,\beta|,\end{aligned}$$

* By uniform convergence for $t \geqslant \epsilon > 0$.

† Cf. DFS, Lemma 4.

‡ L. S. Bosanquet, *Proc. London Math. Soc.* (2) 41 (1936), 517–28, Theorem 1. Cf. DFS, Lemma 5.

§ Cf. DFS, Lemma 6.

by Lemma 3, and similarly,

$$2\sigma_n - \tau_n - \tau_{n-1} = o(1) \quad |C, \beta|.$$

LEMMA 8. *If $\sum u_n$ is summable $|C|$, then a necessary and sufficient condition that $\sum_{n=1}^{\infty} n(u_{n-1} - u_n)$ should be summable $|C, \alpha+1|$ to sum s, where $\alpha > -1$, is that $\sum_{n=0}^{\infty} u_n$ should be summable $|C, \alpha|$ to s.*

The proof is similar to that of the (C) analogue.*

The proof of Theorem 1 is now similar to that of the (C) analogue, but we give the steps for the convenience of the reader.

Proof of Theorem 1. First suppose that $\sum nB_n$ is summable $|C, \alpha+1|$ to s. It follows, by Lemma 4, that $\psi(t) \sim st\,|C|$ as $t \to +0$, and hence, by Lemma 5, that $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable $|CL|$ and tends to $s\ |C|$ as $t \to +0$. Hence, by Lemma 6, its Fourier series, $\frac{1}{2}\alpha(0) + \sum \alpha(n)\cos nt$, is summable $|C|$ for $t = 0$, and, by Lemma 7, $\sum \alpha(n+\frac{1}{2})$ is summable $|C|$. Now $\alpha(n-\frac{1}{2}) - \alpha(n+\frac{1}{2}) = B_n$, and hence by the necessity part of Lemma 8, $\sum \alpha(n+\frac{1}{2})$ is summable $|C, \alpha|$ to s, and so, by Lemma 7, $\frac{1}{2}\alpha(0) + \sum \alpha(n)$ is summable $|C, \alpha|$ to s.

Conversely, if $\psi(t)/(2\sin\frac{1}{2}t)$ is integrable $|CL|$, and its Fourier series $\frac{1}{2}\alpha(0) + \sum \alpha(n)\cos nt$ is summable $|C, \alpha|$ to s at $t = 0$, then, by Lemma 7, $\sum \alpha(n+\frac{1}{2})$ is summable $|C, \alpha|$ to s, and hence, by the sufficiency part of Lemma 8, $\sum nB_n$ is summable $|C, \alpha+1|$ to s.

THEOREM 2. *If $\alpha \geqslant 0$ and $f(t)$ is integrable L, a necessary and sufficient condition that $\sum nB_n$ should be summable $|C, \alpha+1|$ to s is that $\psi(2t)/(2\sin t)$ should be integrable $|CL|$ in $(0, \frac{1}{2}\pi)$ and its Fourier series† summable $|C, \alpha|$ to s at $t = 0$.*

* Cf. DFS, Lemma 7. This result is included in a theorem of H. C. Chow, *J. of London Math. Soc.* 14 (1939), 101–12, Theorem A.

† i.e. $\alpha(\frac{1}{2})\cos t + 0 + \alpha(\frac{3}{2})\cos 3t + 0 + \ldots$, where

$$\alpha(n+\tfrac{1}{2}) = \frac{4}{\pi} \int_{\to 0|C|}^{\frac{1}{2}\pi} \frac{\psi(2t)}{2\sin t}\cos(2n+1)t\,dt.$$

The suppression of a zero term at the beginning of the series is immaterial. For, if one of the series $\sum_{n=0}^{\infty} u_n$ and $0 + \sum_{n=1}^{\infty} u_{n-1}$ is summable $|C, \alpha|$ $(\alpha > -1)$, then so is the other. This follows from the identity $s_n^{\alpha} = \dfrac{n+\alpha+1}{n+1}\sigma_{n+1}^{\alpha}$, where s_n^{α}, σ_n^{α} are the nth Cesàro means of order α of the two series.

The proof is similar to that of Theorem 1, but instead of Lemma 7 we use the following lemma:

LEMMA 9. *If one of the series $u_0+u_1+...$ and $u_0+0+u_1+0+...$ is summable $|C,\alpha|$, where $\alpha \geqslant 0$, then so is the other.**

If we argue with Rieszian means instead of Cesàro means,† the result follows from the identity

$$\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\alpha}u_n = \sum_{2n<\omega'}\left(1-\frac{2n}{\omega'}\right)^{\alpha}u_n,$$

where $\omega' = 2\omega$.

* This result is false if $-1<\alpha<0$. For suppose that $-1<\alpha<\beta<0$ and let $u_n = n^{\beta-1}$. Then $\sum u_n$ is absolutely convergent and $nu_n = o(1)\ |C,0|$. Hence $nu_n = o(1)\ |C,\alpha+1|$, and so, since $\sum u_n$ is summable $|C|$, it is summable $|C,\alpha|$, cf. Bosanquet and Hyslop, loc. cit. On the other hand, if $v_{2n} = u_n$, $v_{2n+1} = 0$, for $n = 0, 1,...,$ then $\sum v_n$ is not summable $|C,\alpha|$. For, if it were, we should have $nv_n = o(1)\ |C,\alpha+1|$, and hence $\sum n^{-\alpha-1}\Delta(nv_n)$ would be absolutely convergent, cf. E. Kogbetliantz, *Bull. des Sc. Math.* (2) 49 (1925), 234–56. This would imply that $\sum n^{-\alpha-1}|nu_n|$ converges, which is not so.

† The equivalence of the $|C,\alpha|$ and $|R,n,\alpha|$ processes has been established by J. M. Hyslop, *Proc. Edinburgh Math. Soc.* (2) 5 (1936), 46–54.

NOTE ON MANIFOLDS

By J. H. C. WHITEHEAD (*Oxford*)

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THE main object of this note is to prove, in a sharpened form, the theorem that some subdivision of any manifold, in the sense of M. H. A. Newman* and J. W. Alexander,† is what S. S. Cairns‡ has called a Brouwer manifold. The words 'element', 'sphere', 'manifold', etc., will have the same meaning, and the same notations will be used, as in Alexander's paper,§ except that $\dot{K}$ will stand for the boundary of K. All our complexes will be 'symbolic', meaning that they are no more than sets of sets (simplexes) of undefined vertices. By a simplex we shall always mean a closed simplex, that is to say, a set of vertices together with all its sub-sets, including the empty set. The latter is the (-1)-simplex, denoted by 1, which is in every complex, except the empty complex 0. The boundary of 1 is 0 and 1 is the boundary of a 0-simplex. Therefore 1 is also a (-1)-sphere. By a 'simplex' we shall always mean an m-simplex, for some finite $m \geqslant 0$. We recall that, if A is a given simplex and b a given vertex, then the elementary subdivision (A, b), applied to a complex $K = AP+Q$, is the subdivision

$$AP+Q \to b\dot{A}P+Q,$$

assuming that $b \not\subset K$. When we write $K = AP+Q$, it is to be understood that $A \not\subset Q$, so that P is the complement of A. We also agree that $P = 0$, $Q = K$ if $A \not\subset K$, in which case (A, b) is applicable to K and leaves it unaltered (it is not applicable to K if $A \subset K$, $b \subset K$). A *stellar subdivision*, always denoted by σ, σ_1, σ', etc., is the resultant of a series‖ of elementary subdivisions. A *partition* πK is a general subdivision,** which can be represented, without further subdivision, as a (rectilinear) partition of any rectilinear, geometrical model of K. We shall always denote partitions by π, π_1, π', etc. It

* M. H. A. Newman, *Proc. Kon. Akad. Amsterdam*, 29 (1926), 611–26.

† J. W. Alexander, *Annals of Math.* 31 (1930), 292–320.

‡ S. S. Cairns, *Proc. National Acad. of Sciences*, 26 (1940), 359–61.

§ J. W. Alexander, loc. cit.

‖ If K is infinite, σK may be a product of infinitely many elementary subdivisions, provided that no simplex is subdivided infinitely many times.

** See M. H. A. Newman, *J. of London Math. Soc.* 2 (1927), 56–64.

is obvious that a stellar subdivision is a partition, also that the resultant of two partitions is a partition. If two complexes are combinatorially equivalent, then some stellar subdivision of a given one of them is a partition of the other.*

Let $\pi_1 A$ and $\pi_2 B$ be partitions, which have no simplex in common, of simplexes A and B. Clearly $(\pi_1 A)^{\cdot}$ and $(\pi_2 B)^{\cdot}$, which we also denote by $\pi_1 \dot{A}$ and $\pi_2 \dot{B}$, are partitions of $\dot{A}$ and $\dot{B}$. Moreover, they may be any partitions, which have no simplex in common. For, given $\pi_1 \dot{A}$ and $\pi_2 \dot{B}$, we may take

$$\pi_1 A = a(\pi_1 \dot{A}), \qquad \pi_2 B = b(\pi_2 \dot{B}),$$

where a and b are new vertices.

LEMMA. *The joins $(\pi_1 A)(\pi_2 B)$ and $(\pi_1 \dot{A})(\pi_2 B)$ are partitions of simplexes, and $(\pi_1 \dot{A})(\pi_2 \dot{B})$ is a partition of the boundary of a simplex.*

If P and Q are given complexes, and if πP is a partition of P which has no simplex in common with Q, it follows by considering the individual simplexes in PQ that $(\pi P)Q = \pi'(PQ)$. Therefore

$$(\pi_1 A)(\pi_2 B) = \pi_1'(A\pi_2 B) = \pi_1'\pi_2'(AB) = \pi_3(AB).$$

Similarly, if $B = bB_1$ (possibly $B_1 = 1$), we have

$$(\pi_1 \dot{A})(\pi_2 B) = \pi_3(\dot{A}B) = \pi_3(b\dot{A}B_1) = \pi_3\sigma(AB_1) = \pi_4(AB_1),$$

where $\sigma = (A, b)$. Finally,

$$(\pi_1 \dot{A})(\pi_2 \dot{B}) = \pi_3(\dot{A}\dot{B}) = \pi_3(\dot{A}B)^{\cdot} = \pi_3[\sigma(AB_1)]^{\cdot} = \pi_4(AB_1)^{\cdot},$$

since $(\sigma K)^{\cdot} = \sigma(\dot{K})$, and the proof is complete.

Let M^n be an n-dimensional manifold, which may be bounded or unbounded, finite or infinite.

THEOREM 1. *There is a stellar subdivision σM^n such that the complement of each simplex† $A \subset \sigma M^n$ is a partition of a simplex or of the boundary of a simplex, according as $A \subset \sigma \dot{M}^n$ or A is inside σM^n.*

Let us describe an element, or sphere, which is a partition of a simplex, or its boundary, as *special*. If the complement of each simplex in M^n is special, there is nothing to prove. Otherwise, let k be the maximum dimensionality of the simplexes in M^n, whose complements are not special. Then $k < n$ (indeed $k < n-1$). Let

* J. H. C. Whitehead, *Proc. Cambridge Phil. Soc.* 31 (1935), 69–75.

† The complement of an n-simplex in M^n is $1 = \pi 1$.

$k < r \leqslant n$ and let $\sigma = (A^r, a)$, where A^r is any r-simplex in M^n and $a \not\subset M^n$. Then, if

$$M^n = A^r P + Q,$$

we have

$$\sigma M^n = a\dot{A}^r P + Q.$$

The simplexes in σM^n which are not in M^n are those of the form aB, where $B \subset \dot{A}^r P$. Let $B = B_1 B_2$, where $B_1 \subset \dot{A}^r$, $B_2 \subset P$ (possibly $B_1 = 1$ or $B_2 = 1$ or both), and let $A^r = A_1 B_1$, $P = B_2 P_1 + P_2$. Then

$$\begin{aligned} \sigma M^n &= a(\dot{A}_1 B_1 + A_1 \dot{B}_1)(B_2 P_1 + P_2) + Q \\ &= aB\dot{A}_1 P_1 + a(A_1 \dot{B}_1 P + \dot{A}_1 B_1 P_2) + Q, \end{aligned}$$

and $aB \not\subset a(A_1 \dot{B}_1 P + \dot{A}_1 B_1 P_2) + Q$ since $a \not\subset Q$, $B_1 \not\subset aA_1 \dot{B}_1 P$, $B_2 \not\subset a\dot{A}_1 B_1 P_2$. Therefore the complement of aB in σM^n is $\dot{A}_1 P_1$. On the other hand,

$$M^n = A^r(B_2 P_1 + P_2) + Q,$$

whence P_1 is the complement of $A^r B_2$ in M^n. Therefore P_1 is special, since $\dim(A^r B_2) \geqslant r > k$, and it follows from the lemma that $\dot{A}_1 P_1$ is special. Thus the complement of each new simplex which is introduced by the subdivision (A^r, a) is special. Let B be any simplex in M^n which is also in σM^n (i.e. $B \subset M^n$, $A^r \not\subset B$) and let

$$M^n = BU + V.$$

If $B \not\subset A^r$, so that $A^r \subset U$ or $A^r \subset V$, it is clear that the complement of B in σM^n is σU. Otherwise let $A^r = BB_1$ and let

$$U = B_1 U_1 + V_1.$$

Then

$$\begin{aligned} \sigma M^n &= \sigma(BB_1 U_1 + BV_1 + V) \\ &= a(B\dot{B}_1 + \dot{B}B_1)U_1 + BV_1 + V \\ &= B(a\dot{B}_1 U_1 + V_1) + a\dot{B}B_1 U_1 + V \\ &= B(\sigma' U) + V_2 \quad (B \not\subset V_2), \end{aligned}$$

where $\sigma' = (B_1, a)$. Therefore, in each case, the complement of B in σM_n is a stellar subdivision of U, and is special if U is special. Therefore, except possibly for the simplexes in M^n whose complements in M^n are not special, the complement in σM^n of any simplex in σM^n is special.

Let $A_1^k, A_2^k, \ldots \subset M^n$ be the k-simplexes whose complements in M^n are not special. Let U be the complement of A_1^k in M^n, let B_1 be any simplex in U, and let $\sigma_0' = (B_1, a)$, $\sigma_0 = (A_1^k B_1, a)$, where $a \not\subset M^n$. Then it follows from the argument above that the complement of A_1^k in $\sigma_0 M^n$ is $\sigma_0' U$. Since U is combinatorially equivalent to a

simplex or its boundary, some stellar subdivision $\sigma_1' U$ is special, and we may obviously assume that none of the vertices introduced by σ_1' is already in M^n. Then $\sigma_1' U$ is the complement of A_1^k in a stellar subdivision $\sigma_1 M^n$, where σ_1 is the resultant of elementary subdivisions of the form $(A_1^k B, b)$, with $B \neq 1$, whence $\dim(A_1^k B) > k$. Since $\dim(A_1^k B) > k$, the complements, in $\sigma_1 M^n$, of all the simplexes in $\sigma_1 M^n$ are special, excepting possibly the simplexes in M^n, other than A_1^k, whose complements in M^n are not special. On repeating this argument it follows that there is a sequence of stellar subdivisions, whose resultant is a subdivision* $\sigma^{(k)} M^n$, in which the complement of each m-simplex is special for $m = k, \ldots, n$, and the theorem follows from induction on k.

If P is the complement of a simplex $A \subset M^n$, then AP is called the star of A. If P is special, so is AP, according to the lemma. Therefore we have the corollary:

Corollary. *There is a stellar subdivision of M^n in which the star of each simplex is a partition of a simplex.*

Theorem 2. *If the complement of each simplex in M^n is a partition of a simplex, or of the boundary of a simplex, then the same is true of any stellar subdivision of M^n.*

This follows from the first part of the proof of Theorem 1, with $r \geqslant 0$.

* If M^n is infinite and the simplexes $A_1^k, A_2^k, \ldots$ are infinite in number, then $\sigma^{(k)}$ will be a product of infinitely many subdivisions of the form $(A_i^k B, b)$ $(i = 1, 2, \ldots)$.

ON THE SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF THE DIRICHLET AND NEUMANN TYPE

By T. LEWIS (*Aberystwyth*)

[Received 14 January 1941]

In a recent note Milne-Thomson (**1**) has shown how to write down the complex potential for any system of line sources, vortices, etc., in the presence of a circular cylinder, and hence, by transformation, in the presence of a cylinder of arbitrary cross-section. The object of the present note is to show how to write down the complete potential for a cylinder of arbitrary cross-section when the stream function is arbitrarily prescribed on its boundary and vanishes at infinity. This covers all the cases considered by Rosa-Morris (**3**) in a series of papers and, by a slight adjustment, is applicable to electrostatic and other problems of the Dirichlet and Neumann type. We shall also deduce a formula which is analogous to a formula given by Schwarz (**2**) for the internal problem in the case of a circle.

Let the transformation

$$z = \zeta \sum_{n=0}^{\infty} A_n \zeta^{-n} \tag{1}$$

map the z-plane on the ζ-plane so that the closed curve C in the z-plane becomes the unit circle Γ ($|\zeta| = 1$) in the ζ-plane, and all the singularities of the transformation lie inside the circle. On the curve C the stream function ψ must be of the form

$$\Psi_c(\zeta_0) = \sum_{-\infty}^{\infty} a_n \zeta_0^{-n}, \tag{2}$$

where $a_{-n} = \bar{a}_n$, a_0 is real, and ζ_0 is written for ζ on the unit circle, so that $\bar{\zeta}_0 = \zeta_0^{-1}$. (A bar over a symbol transforms it into its conjugate complex.)

The problem is to find the complex potential,

$$w = \phi + i\psi,$$

when Ψ_c is arbitrarily prescribed and w approaches a constant at infinity.

We see from (2) that

$$\Psi_c = \mathrm{R}\Big[a_0 + 2\sum_{n=1}^{\infty} a_n \zeta_0^{-n}\Big]. \tag{3}$$

It follows immediately that

$$w = ia_0 + 2i\sum_{n=1}^{\infty} a_n \zeta^{-n} \tag{4}$$

satisfies all the conditions of the problem. The constant term ia_0 can be ignored, because in general it has no physical significance.

Again, multiplying both sides of (2) by $\zeta_0^{n-1}\,d\zeta_0$ and integrating round the unit circle, we get

$$\oint_\Gamma \Psi_c \zeta_0^{n-1}\,d\zeta_0 = a_n \oint_\Gamma \frac{d\zeta_0}{\zeta_0} = i2\pi a_n. \tag{5}$$

Substituting for a_n in (4) we get

$$w = \frac{1}{2\pi}\oint_\Gamma \Psi_c \frac{d\zeta_0}{\zeta_0} + \frac{1}{\pi}\oint_\Gamma \Psi_c \zeta^{-1} \sum_{n=0}^{\infty}\left(\frac{\zeta_0}{\zeta}\right)^n d\zeta_0.$$

Ignoring the constant first term we have

$$w = \frac{1}{\pi}\oint_\Gamma \frac{\Psi_c(\zeta_0)\,d\zeta_0}{\zeta-\zeta_0}. \tag{6}$$

This formula is similar to the formula given by Schwarz for the internal problem.

Circulation round the cylinder is taken into account by adding to w a term $i\kappa\log\zeta$.

To illustrate the application of the result let us consider the cylinder in the presence of a line vortex at $z = z_1$, the liquid being at rest at infinity. The undisturbed potential due to the vortex is

$$i\kappa\log(z-z_1).$$

In the neighbourhood of the vortex this function behaves like

$$w_0 = i\kappa\log(\zeta-\zeta_1) + \text{constant},$$

where $\zeta = \zeta_1$ is the transform of $z = z_1$. Also at infinity $\log(z-z_1)$ behaves like $\log(\zeta-\zeta_1)$.

Let the complex potential be

$$w = w_0 + w_1,$$

where w_1 represents the disturbance due to the presence of the cylinder. At the cylinder the part of the stream function due to the cylinder is

$$\begin{aligned}\psi_1 = -\psi_0 &= -\frac{1}{2i}(w_0-\bar{w}_0)\\ &= -\tfrac{1}{2}\kappa\{\log(\zeta_0-\zeta_1)+\log(\zeta_0^{-1}-\bar{\zeta}_1)\}\\ &= -\mathrm{R}[\kappa\log(\zeta_0^{-1}-\bar{\zeta}_1)].\end{aligned}$$

It follows immediately that

$$w_1 = -i\kappa\log(\zeta^{-1}-\bar{\zeta}_1)$$

and $$w = i\kappa\log\{(\zeta-\zeta_1)/(\zeta^{-1}-\bar{\zeta}_1)\}.$$

Again, let us consider the case of a cylinder in the presence of a doublet at $z = z_1$. The undisturbed potential is

$$\mu/(z-z_1).$$

In the neighbourhood of the doublet this behaves like

$$w_0 = \mu_1/(\zeta-\zeta_1),$$

where $$\mu_1 = \mu\Big/\left(\frac{dz}{d\zeta}\right)_{\zeta=\zeta_1}.$$

Proceeding as before, we put

$$w = w_0+w_1.$$

At the cylinder

$$\psi_1 = -\psi_0 = -\frac{1}{2i}(w_0-\bar{w}_0) = -\frac{1}{2i}\left(\frac{\mu_1}{\zeta_0-\zeta_1}-\frac{\bar{\mu}_1}{\zeta_0^{-1}-\bar{\zeta}_1}\right)$$

$$= -\mathrm{R}\left[\frac{i\bar{\mu}_1}{\zeta_0^{-1}-\bar{\zeta}_1}\right].$$

It follows that $$w_1 = \bar{\mu}_1/(\zeta^{-1}-\bar{\zeta}_1),$$

and $$w = \mu_1/(\zeta-\zeta_1)+\bar{\mu}_1/(\zeta^{-1}-\bar{\zeta}_1).$$

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ON EXPANSIONS IN EIGENFUNCTIONS (IV)

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1. In the previous papers* the eigenfunction expansions arising from certain differential equations were determined. A general theory of these expansions has been given by Weyl,† who deduces them from the corresponding results in the theory of integral equations. My next object is to prove Weyl's main theorems independently of the theory of integral equations. In the present paper I consider the case where the expansion is a series, and no integral term occurs.

2. For the sake of completeness we shall first prove the main result of Kap. 1 of Weyl's first paper.

It is a question of extending to an infinite interval, or to an interval ending at a singularity, the ordinary Sturm-Liouville expansion for a finite interval. Let L denote the operator‡

$$\frac{d^2}{dx^2}-q(x), \tag{2.1}$$

where $q(x)$ is continuous for $0 \leqslant x < \infty$. If $\phi(x)$, $\psi(x)$, $L(\phi)$, and $L(\psi)$ are continuous,

$$\int\limits_0^a \psi L(\phi)\,dx = \big[\psi(x)\phi'(x)\big]_0^a - \int\limits_0^a \{\phi'(x)\psi'(x)-q(x)\phi(x)\psi(x)\}\,dx.$$

From the symmetry of the last integral it is clear that

$$\int\limits_0^a \{\psi L(\phi)-\phi L(\psi)\}\,dx = \big[W_x(\psi,\phi)\big]_0^a, \tag{2.2}$$

where $W(\psi,\phi)$ or $W_x(\psi,\phi)$ denotes the Wronskian

$$\psi(x)\phi'(x)-\phi(x)\psi'(x).$$

If ψ satisfies the differential equation

$$(L-w)\psi = 0, \tag{2.3}$$

* E. C. Titchmarsh, *J. of London Math. Soc.* 14 (1939), 274–8, and *Quart. J. of Math.* (Oxford), 11 (1940), 129–40 and 141–5.

† H. Weyl, *Math. Annalen*, 68 (1910), 220–69; *Göttinger Nachrichten* (1910), 442–67.

‡ The argument could easily be extended to cover Weyl's case

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right)-q(x).$$

where $w = u+iv$, $v \neq 0$, then $\bar{\psi}$ satisfies the conjugate equation. Hence (2.2) gives

$$\int_0^a (\psi\bar{w}\bar{\psi}-\bar{\psi}w\psi)\,dx = \left[W_x(\psi,\bar{\psi})\right]_0^a,$$

i.e.
$$2v\int_0^a |\psi|^2\,dx = iW_a(\psi,\bar{\psi})-iW_0(\psi,\bar{\psi}). \tag{2.4}$$

If ϕ and ψ both satisfy (2.3), then (2.2) shows that $W_x(\phi,\psi)$ is a constant.

Now let $f(x) = f(x,w)$, $F(x) = F(x,w)$ be the solutions of (2.3) which satisfy the boundary conditions

$$f(0) = \sin h,\quad f'(0) = -\cos h,\quad F(0) = \cos h,\quad F'(0) = \sin h, \tag{2.5}$$

where h is real. It is known that these solutions exist, and are integral functions of w for each x. Also

$$W_x(f,F) = W_0(f,F) = \sin^2 h+\cos^2 h = 1. \tag{2.6}$$

The general solution of (2.3) is

$$F(x)+lf(x),$$

where l is any real or complex number. We now consider those solutions which satisfy a real boundary condition at $x = a$, i.e. a condition of the form

$$\{F(a)+lf(a)\}\cos j+\{F'(a)+lf'(a)\}\sin j = 0, \tag{2.7}$$

where j is real. This gives

$$l = -\frac{F(a)\cot j+F'(a)}{f(a)\cot j+f'(a)}. \tag{2.8}$$

For each a, as $\cot j$ varies, l describes a circle in the complex plane, say C_a. If we replace $\cot j$ by a complex variable z, $l = \infty$ corresponds to $z = -f'(a)/f(a)$. Hence the centre of C_a corresponds to $z = -\bar{f}'(a)/\bar{f}(a)$, i.e. it is

$$l_a = -W_a(F,\bar{f})/W_a(f,\bar{f}).$$

Since $-F'(a)/f'(a)$ is on C_a, the radius r_a of C_a is

$$r_a = \left|\frac{F'(a)}{f'(a)}-\frac{W_a(F,\bar{f})}{W_a(f,\bar{f})}\right| = \left|\frac{W_a(F,f)}{W_a(f,\bar{f})}\right| = \left\{2v\int_0^a |f(x)|^2\,dx\right\}^{-1}. \tag{2.9}$$

The interior of C_a corresponds to the upper half of the z-plane if

$$\mathbf{I}\{-\bar{f}'(a)/\bar{f}(a)\} > 0$$

or if

$$iW_a(f,\bar{f}) > 0,$$

which is true if $v > 0$. The interior of C_a is then given by

$$i\left\{-\frac{F'(a)+lf'(a)}{F(a)+lf(a)}+\frac{\bar{F}'(a)+\bar{l}\bar{f}'(a)}{\bar{F}(a)+\bar{l}\bar{f}(a)}\right\} < 0,$$

i.e. $$i|l|^2W_a(f,\bar{f})+ilW_a(f,\bar{F})+i\bar{l}W_a(F,\bar{f})+iW_a(F,\bar{F}) < 0.$$

Using (2.4) with $\psi = F+lf$, this reduces to

$$\int_0^a |F(x)+lf(x)|^2\,dx < \mathbf{I}(l)/v. \tag{2.10}$$

The same result is obtained if $v < 0$; $\mathbf{I}(l)$ therefore has the same sign as v.

It follows that, if l is inside or on C_a, and $0 < a' < a$, then

$$\int_0^{a'} |F+lf|^2\,dx \leqslant \int_0^a |F+lf|^2\,dx \leqslant \mathbf{I}(l)/v,$$

so that l is inside or on $C_{a'}$. Hence $C_{a'}$ includes C_a if $a' < a$. Hence as $a \to \infty$ the circles C_a converge either to a limit-circle or to a limit-point.

If l_1 is the limit-point, or any point on the limit-circle, (2.10), with l_1 for l and '$\leqslant$' for '$<$', holds for all values of a. Hence

$$\int_0^\infty |F(x)+l_1f(x)|^2\,dx \leqslant \mathbf{I}(l_1)/v. \tag{2.11}$$

Hence (2.3) *has a solution belonging to* $L^2(0,\infty)$.

In the limit-circle case, r_a tends to a positive limit as $a \to \infty$, so that

$$\int_0^\infty |f(x)|^2\,dx < \infty.$$

In this case *every solution of* (2.3) *belongs to* $L^2(0,\infty)$.

Let us now write

$$g(x) = g(x,w) = F(x)+l_1f(x).$$

Then by (2.7), if $\mathbf{I}(w) \neq 0$, $\mathbf{I}(w') \neq 0$,

$$\begin{aligned}0 &= W_a[g(x,w)+\{l(w)-l_1(w)\}f(x,w),\,g(x,w')+\{l(w')-l_1(w')\}f(x,w')]\\ &= W_a\{g(x,w),g(x,w')\}+\{l(w)-l_1(w)\}W_a\{f(x,w),g(x,w')\}+\\ &\qquad +\{l(w')-l_1(w')\}W_a\{g(x,w),f(x,w')\}+\\ &\qquad +\{l(w)-l_1(w)\}\{l(w')-l_1(w')\}W_a\{f(x,w),f(x,w')\}.\end{aligned}$$

Now

$$W_a\{f(x,w),g(x,w')\} = (w'-w)\int\limits_0^a f(x,w)g(x,w')\,dx + W_0\{f(x,w),g(x,w')\}$$

$$= O\left\{\int\limits_0^a |f(x,w)|^2\,dx\right\}^{\frac{1}{2}},$$

as $a \to \infty$, w and w' being fixed. Also in the limit-point case

$$|l(w)-l_1(w)| \leqslant 2r_a = \left\{v\int\limits_0^a |f(x,w)|^2\,dx\right\}^{-1}, \tag{2.12}$$

so that $$\lim_{a\to\infty}\{l(w)-l_1(w)\}W_a\{f(x,w),g(x,w')\} = 0.$$

This also holds in the limit-circle case if $l(w) \to l_1(w)$, since then the second factor is bounded. Similar arguments apply to the other terms. Hence

$$\lim_{a\to\infty} W_a\{g(x,w),g(x,w')\} = 0. \tag{2.13}$$

The case $w' = \bar{w}$ of this result, combined with (2.4) with $\psi = g(x,w)$, shows that actually the case of equality holds in (2.11).

3. To obtain the expansion of an arbitrary function $\psi(x)$ in terms of eigenfunctions, we consider, as in the previous papers, the solution $\psi(x,t)$ of

$$L\psi = i\frac{\partial\psi}{\partial t} \tag{3.1}$$

such that $\psi(x,0) = \psi(x)$, where $\psi(x)$ belongs to $L^2(0,\infty)$, and

$$\psi(0,t)\cos h + \psi_x(0,t)\sin h = 0 \tag{3.2}$$

for all $t > 0$. As before, if

$$\Psi_+(x,w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \psi(x,t)e^{iwt}\,dt, \tag{3.3}$$

and $\Psi_-(x,w)$ is the corresponding integral over $(-\infty,0)$, these conditions are formally equivalent to

$$(L-w)\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}\psi(x) \tag{3.4}$$

and $$\Psi_+(0,w)\cos h + \Psi_{+,x}(0,w)\sin h = 0, \tag{3.5}$$

with conjugate conditions for $\Psi_-(x,w)$. The general solution of (3.4) is

$$\Psi_+(x,w) = Af(x,w)+Bg(x,w)-$$

$$-\frac{i}{\sqrt{(2\pi)}}\left\{g(x,w)\int\limits_0^x f(y,w)\psi(y)\,dy + f(x,w)\int\limits_x^\infty g(y,w)\psi(y)\,dy\right\}. \tag{3.6}$$

The boundary condition (3.5) gives $B = 0$. We also take $A = 0$. In the limit-point case this is necessary to make Ψ_+ belong to $L^2(0,\infty)$. In the limit-circle case Ψ_+ belongs to L^2 for all A, but $A = 0$ is the only value which leads to (3.9).

We also write this as

$$\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}\int_0^\infty G(x,y)\psi(y)\,dy,$$

where $$G(x,y) = G(x,y,w) = \begin{cases} g(x,w)f(y,w) & (y < x), \\ f(x,w)g(y,w) & (y > x). \end{cases}$$

Suppose in the first place that $\psi(y) = 0$ for $y > \Delta$. Then for $x > \Delta$

$$\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}g(x,w)\int_0^\Delta f(y,w)\psi(y)\,dy.$$

Hence, by (2.13),

$$\lim_{x\to\infty} W(\Psi_+,\overline{\Psi}_+) = \lim_{x\to\infty} KW(g,\bar{g}) = 0. \tag{3.7}$$

Now, by (2.2), (3.4), (3.5),

$$\int_0^a \left\{\Psi_+\left(\bar{w}\overline{\Psi}_+ + \frac{i\psi}{\sqrt{(2\pi)}}\right) - \overline{\Psi}_+\left(w\Psi_+ - \frac{i\psi}{\sqrt{(2\pi)}}\right)\right\} dx = W_a(\Psi_+,\overline{\Psi}_+).$$

Hence

$$2v\int_0^a |\Psi_+|^2\,dx = \frac{2}{\sqrt{(2\pi)}}\int_0^a \mathbf{R}(\Psi_+)\psi\,dx + iW_a(\Psi_+,\overline{\Psi}_+)$$

$$\leqslant \frac{2}{\sqrt{(2\pi)}}\left\{\int_0^a |\Psi_+|^2\,dx\int_0^a |\psi|^2\,dx\right\}^{\frac{1}{2}} + |W_a(\Psi_+,\overline{\Psi}_+)|. \tag{3.8}$$

If $\left(\int_0^a |\Psi_+|^2\,dx\right)^{\frac{1}{2}} > 0$ for $a > a_0$, we have, on dividing by this factor and making $a \to \infty$,

$$2v\left(\int_0^\infty |\Psi_+|^2\,dx\right)^{\frac{1}{2}} \leqslant \frac{2}{\sqrt{(2\pi)}}\left(\int_0^\infty \{\psi(x)\}^2\,dx\right)^{\frac{1}{2}},$$

i.e. $$\int_0^\infty |\Psi_+|^2\,dx \leqslant \frac{1}{2\pi v^2}\int_0^\infty \{\psi(x)\}^2\,dx. \tag{3.9}$$

Since the alternative is the vanishing of the left-hand side, (3.9) holds in any case.

Now taking any ψ of L^2, and distinguishing the function with $\psi(y) = 0$ for $y > \Delta$ by a suffix Δ, $\Psi_{+,\Delta} \to \Psi_+$ uniformly over any finite x-interval. Also

$$\int_0^a |\Psi_{+,\Delta}|^2\,dx \leqslant \int_0^\infty |\Psi_{+,\Delta}|^2\,dx \leqslant \frac{1}{2\pi v^2}\int_0^\Delta \psi^2\,dx \leqslant \frac{1}{2\pi v^2}\int_0^\infty \psi^2\,dx.$$

Hence, making $\Delta \to \infty$ and then $a \to \infty$, (3.9) is proved for all $\psi(x)$ of $L^2(0,\infty)$.

Another important relation is

$$(w'-w)\int_0^\infty G(x,y,w)\Psi_+(y,w')\,dy = \Psi_+(x,w')-\Psi_+(x,w). \tag{3.10}$$

By (2.2),

$$\int_0^\infty \{G(x,y,w)L\Psi_+(y,w')-\Psi_+(y,w')LG(x,y,w)\}\,dy$$
$$= [W\{G(x,y,w),\Psi_+(y,w')\}]_0^x+[...]_x^\infty.$$

The Wronskian vanishes at the origin, by (3.5), and at infinity if $\psi(x) = 0$ for $x > \Delta$, by (2.13). Hence the right-hand side is

$$g(x,w)W\{f(x,w),\Psi_+(x,w')\}-f(x,w)W\{g(x,w),\Psi_+(x,w')\} = \Psi_+(x,w').$$

The left-hand side is

$$\int_0^\infty \left[G(x,y,w)\left\{w'\Psi_+(y,w')-\frac{i}{\sqrt{(2\pi)}}\psi(y)\right\}-\Psi_+(y,w')wG(x,y,w)\right]dy,$$

and (3.10) follows. The result also holds for any ψ of $L^2(0,\infty)$, since, by (3.9), $\Psi_{+,\Delta}$ converges in mean to Ψ_+, and so each term of (3.10) with finite Δ converges to the corresponding term with $\Delta = \infty$.

4. We shall require the following lemmas.

LEMMA α. *Let $\phi(w)$ be regular and*

$$|\phi(w)| \leqslant M/|v|$$

for $-r \leqslant u \leqslant r$; $-r \leqslant v \leqslant r$. Then

$$|\phi(w)| \leqslant 3M/r \quad (u = 0;\ -r \leqslant v \leqslant r).$$

This was stated without proof in paper III.* Consider

$$\chi(w) = (w^2-r^2)\phi(w).$$

* E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 11 (1940), 145.

On the top and bottom sides of the square

$$|\chi(w)| \leqslant (|w|^2+r^2)M/r \leqslant 3rM.$$

On the left and right sides

$$|\chi(w)| \leqslant |v|(|w|+r)M/|v| \leqslant 3rM.$$

Hence $|\chi(w)| \leqslant 3rM$ throughout the square. Hence on the imaginary axis

$$|\phi(w)| \leqslant \frac{3rM}{|w^2-r^2|} = \frac{3rM}{v^2+r^2} \leqslant \frac{3M}{r}.$$

Lemma β. *For all functions ψ and χ of $L^2(0,\infty)$*

$$\int_0^\infty \chi(x)\,dx \int_0^\infty \psi(y)G(x,y,w)\,dy = \int_0^\infty \psi(y)\,dy \int_0^\infty \chi(x)G(x,y,w)\,dx.$$

If x is restricted to a finite range $(0,X)$, the inversion is justified by uniform convergence. It is therefore sufficient to prove that

$$\lim_{X\to\infty} \int_0^\infty \psi(y)\,dy \int_X^\infty \chi(x)G(x,y,w)\,dx = 0.$$

By (3.9), $$\int_0^\infty dy \left| \int_X^\infty \chi(x)G(x,y,w)\,dx \right|^2 \leqslant \frac{1}{v^2} \int_X^\infty \chi^2\,dx.$$

The result therefore follows from Schwarz's inequality.

Lemma γ. *Let $f_n(x)$ be a sequence of functions which converges in mean to $f(x)$ over any finite interval, while*

$$\int_0^\infty |f_n(x)|^2\,dx \leqslant K$$

for all n. Then, if $g(x)$ belongs to $L^2(0,\infty)$,

$$\lim_{n\to\infty} \int_0^\infty f_n(x)g(x)\,dx = \int_0^\infty f(x)g(x)\,dx.$$

We have $$\int_0^X |f(x)|^2\,dx = \lim_{n\to\infty} \int_0^X |f_n(x)|^2\,dx \leqslant K$$

for every X, so that $f(x)$ is $L^2(0,\infty)$. Now

$$\left|\int_0^\infty (f-f_n)g\,dx\right| \leqslant \left|\int_0^X\right| + \left|\int_X^\infty\right| \leqslant \left\{\int_0^X |f-f_n|^2\,dx \int_0^\infty |g|^2\,dx\right\}^{\frac{1}{2}} +$$
$$+ \left\{\int_0^\infty |f-f_n|^2\,dx \int_X^\infty |g|^2\,dx\right\}^{\frac{1}{2}}.$$

The second term can be made less than any given ϵ for all n by choice of X. Having fixed X, the first term tends to zero as $n \to \infty$, and the result follows.

5. For a given j the $l = l(a, w)$ defined by (2.8) is an analytic function of w, regular and bounded for $\mathbf{I}(w) \geqslant \delta > 0$ (or $\leqslant -\delta$) as $a \to \infty$. Hence by Vitali's theorem there is a sequence of values of a, say a_ν, such that $l(a_\nu, w)$ converges to a limit $l_1(w)$, which is an analytic function of w, regular in the upper (or lower) half-plane. In the limit-point case $l_1(w)$ is the limit-point. In the limit-circle case $l_1(w)$ is a point on the limit-circle, which may or may not depend on j.

The function $\Psi_+(x, w)$ is therefore for each x an analytic function of w, regular in the upper half-plane. Similarly $\Psi_-(x, w)$ is analytic and regular in the lower half-plane.

We shall now suppose that Ψ_+ and Ψ_- are analytic continuations of each other, their only singularities being simple poles $w_1, w_2, \ldots$ on the real axis, the poles (not the residues) being independent of x. We shall consider later cases in which this is actually true.

Let the residue of $\Psi_+(x, w)$ at w_n be $\phi_n(x)$. Then

$$\phi_n(x) = \frac{1}{2\pi i} \int_\gamma \Psi_+(x, w)\, dw, \tag{5.1}$$

where γ is a contour surrounding w_n, but excluding the other poles.

We have formally

$$(L - w_n)\phi_n(x) = \frac{1}{2\pi i} \int_\gamma (L - w_n)\Psi_+(x, w)\, dw$$

$$= \frac{1}{2\pi i} \int_\gamma \left\{(w - w_n)\Psi_+(x, w) - \frac{i}{\sqrt{(2\pi)}}\psi(x)\right\} dw = 0. \tag{5.2}$$

To justify this we have

$$\frac{d^2}{dx^2}\left\{\Psi_+(x, w) + \frac{i}{\sqrt{(2\pi)}}\,\omega(x)\right\} = \{q(x) + w\}\Psi_+(x, w),$$

where $\omega(x)$ is the integral of the integral of $\psi(x)$. Now the argument of § 2 of the previous paper (e.g. with $\xi = x + 1$) shows that $v\Psi_+(x, w)$ is bounded if x and u lie in finite intervals. Hence, by Lemma α,

$\Psi_+(x,w)$ is bounded for w on γ and x in a finite interval (taking γ to be a small rectangle). Hence the process

$$\frac{d^2}{dx^2}\int\limits_\gamma \left\{\Psi_+(x,w)+\frac{i}{\sqrt{(2\pi)}}\omega(x)\right\} dw = \int\limits_\gamma \{q(x)+w\}\Psi_+(x,w)\,dw$$

is justified, and the result is equivalent to (5.2).

Also, by (5.1) and (3.5),

$$\phi_n(0)\cos h+\phi_n'(0)\sin h = 0. \tag{5.3}$$

Again,
$$\phi_n(x) = \lim_{v\to 0} iv\Psi_+(x,w_n+iv), \tag{5.4}$$

and the right-hand side is bounded for $0 \leqslant x \leqslant a$. Hence

$$\int\limits_0^a |\phi_n(x)|^2\,dx = \lim_{v\to 0}\int\limits_0^a v^2|\Psi_+(x,w_n+iv)|^2\,dx \leqslant \frac{1}{2\pi}\int\limits_0^\infty \psi^2\,dx. \tag{5.5}$$

Hence $\phi_n(x)$ belongs to $L^2(0,\infty)$.

Putting $w' = w_n+iv'$ in (3.10), multiplying by iv', and making $v' \to 0$, we obtain

$$(w_n-w)\int\limits_0^\infty G(x,y)\phi_n(y)\,dy = \phi_n(x), \tag{5.6}$$

the process being justified by (3.9) and Lemma γ. Hence

$$(w_n-w)\int\limits_0^\infty\int\limits_0^\infty G(x,y)\phi_m(x)\phi_n(y)\,dxdy = \int\limits_0^\infty \phi_m(x)\phi_n(x)\,dx,$$

where by Lemma β the integral on the left can be evaluated in either order. Interchanging m and n and subtracting, it follows that the integral on the left vanishes, and hence so does that on the right. Hence the $\phi_n(x)$ form an orthogonal set.

Let
$$\chi_n(x) = \frac{\phi_n(x)}{\phi_n(0)\sin h-\phi_n'(0)\cos h}.$$

Then, by (5.3),
$$\chi_n(0) = \sin h, \qquad \chi_n'(0) = -\cos h.$$

Hence $\chi_n(x) = f(x,w_n)$, i.e.

$$\phi_n(x) = \{\phi_n(0)\sin h-\phi_n'(0)\cos h\}f(x,w_n). \tag{5.7}$$

Let
$$\psi_n(x) = f(x,w_n)\left[\int\limits_0^\infty \{f(y,w_n)\}^2\,dy\right]^{-\frac{1}{2}}. \tag{5.8}$$

Then the $\psi_n(x)$ form a normal orthogonal set, and are independent of $\psi(x)$.

Again, multiplying (5.6) by $-i(2\pi)^{-\frac{1}{2}}\psi(x)$ and integrating, and using Lemma β,

$$(w_n-w)\int_0^\infty \phi_n(y)\Psi_+(y,w)\,dy = -\frac{i}{\sqrt{(2\pi)}}\int_0^\infty \phi_n(x)\psi(x)\,dx. \tag{5.9}$$

Taking $w = w_n+iv$, making $v \to 0$, and using Lemma γ,

$$\int_0^\infty \{\phi_n(y)\}^2\,dy = \frac{i}{\sqrt{(2\pi)}}\int_0^\infty \phi_n(x)\psi(x)\,dx. \tag{5.10}$$

Writing
$$\phi_n(x) = \frac{i}{\sqrt{(2\pi)}}c_n\psi_n(x),$$

this gives
$$\int_0^\infty \psi_n(x)\psi(x)\,dx = c_n\int_0^\infty \{\psi_n(x)\}^2\,dx = c_n.$$

Hence
$$\phi_n(x) = \frac{i}{\sqrt{(2\pi)}}\psi_n(x)\int_0^\infty \psi_n(y)\psi(y)\,dy. \tag{5.11}$$

6. Now suppose that ψ and $L\psi$ are $L^2(0,\infty)$, and that

$$\psi(0)\cos h+\psi'(0)\sin h = 0, \tag{6.1}$$

$$\lim_{x\to\infty} W\{g(x,w),\psi(x)\} = 0 \tag{6.2}$$

for all complex w. Then by (2.2)

$$\int_0^x f(y,w)\psi(y)\,dy = \frac{1}{w}\int_0^x Lf(y,w)\psi(y)\,dy$$

$$= \frac{1}{w}\int_0^x f(y,w)\,L\psi(y)\,dy+\frac{1}{w}W_x(\psi,f)$$

since $W_0(\psi,f) = 0$. Similarly,

$$\int_x^\infty g(y,w)\psi(y)\,dy = \frac{1}{w}\int_x^\infty g(y,w)L\psi(y)\,dy-\frac{1}{w}W_x(\psi,g).$$

Since $$fW(\psi,g)-gW(\psi,f) = \psi W(f,g) = \psi,$$
we obtain

$$\Psi_+(x,w) = \frac{i\psi(x)}{w\sqrt{(2\pi)}} -$$
$$-\frac{i}{w\sqrt{(2\pi)}}\left\{g(x,w)\int_0^x f(y,w)L\psi(y)\,dy + f(x,w)\int_x^\infty g(y,w)L\psi(y)\,dy\right\}. \quad (6.3)$$

The last term involves $L\psi$ in the same way that Ψ_+ involves ψ. Hence, if

$$\Phi(w) = \int_0^\infty \psi(x)\Psi_+(x,w)\,dx, \quad (6.4)$$

we obtain from (3.9) and Schwarz's inequality

$$\Phi(w) = \frac{i}{w\sqrt{(2\pi)}}\int_0^\infty \{\psi(x)\}^2\,dx + O\left(\frac{1}{|wv|}\right). \quad (6.5)$$

Hence the usual integration round a large semicircle gives

$$\lim_{R\to\infty}\int_{-R}^{R} \Phi(u+iv)\,du = \sqrt{(\tfrac{1}{2}\pi)}\int_0^\infty \{\psi(x)\}^2\,dx. \quad (6.6)$$

Now, by (5.8),

$$\mathbf{R}\Phi(w) = \sqrt{(2\pi)}v\int_0^\infty |\Psi_+|^2\,dx \geqslant 0 \quad (v>0). \quad (6.7)$$

Hence, taking the real part of (6.6),

$$\int_{-\infty}^{\infty} \mathbf{R}\Phi(u+iv)\,du = \sqrt{(\tfrac{1}{2}\pi)}\int_0^\infty \{\psi(x)\}^2\,dx \quad (6.8)$$

for every positive v.

The function $\Phi(w)$ is regular except at the points w_n, where it has simple poles with residues

$$\int_0^\infty \psi(x)\phi_n(x)\,dx = \frac{i}{\sqrt{(2\pi)}}c_n^2.$$

For $$(w-w_1)...(w-w_n)\int_0^X \psi(x)\Psi_+(x,w)\,dx$$

is regular and bounded for $|w| \leqslant |w_n|$, and converges uniformly as

$X \to \infty$ in any part of this circle in the upper (or lower) half-plane; and similarly

$$\int_0^\infty \psi(x)\left\{\Psi_+(x,w)-\frac{\phi_n(x)}{w-w_n}\right\} dx$$

is regular at w_n.

If α and β are not poles of $\Phi(w)$,

$$\sum_{\alpha<w_n<\beta} c_n^2 = -\frac{1}{\sqrt{(2\pi)}}\left(\int_{\alpha-i\delta}^{\beta-i\delta} - \int_{\alpha+i\delta}^{\beta+i\delta} + \int_{\beta-i\delta}^{\beta+i\delta} - \int_{\alpha-i\delta}^{\alpha+i\delta}\right)\Phi(w)\, dw. \tag{6.9}$$

Since $\Phi(w)$ is purely imaginary on the real axis,

$$\Phi(u-iv) = -\bar{\Phi}(u+iv).$$

Hence the first two terms together give

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_\alpha^\beta \mathbf{R}\Phi(u+i\delta)\, du.$$

Hence

$$\sum_{\alpha<w_n<\beta} c_n^2 = \sqrt{\left(\frac{2}{\pi}\right)} \lim_{\delta\to 0} \int_\alpha^\beta \mathbf{R}\Phi(u+i\delta)\, du \leqslant \int_0^\infty \{\psi(x)\}^2\, dx$$

by (6.7) and (6.8). Hence $\sum c_n^2$ is convergent, and

$$\sum_{n=1}^\infty c_n^2 \leqslant \int_0^\infty \{\psi(x)\}^2\, dx. \tag{6.10}$$

We have to prove that this is actually an equality. Now (6.9) gives

$$\sum_{\alpha<w_n<\beta} c_n^2 = \sqrt{\left(\frac{2}{\pi}\right)} \int_\alpha^\beta \mathbf{R}\Phi(u+i\delta)\, du + \frac{1}{\sqrt{(2\pi)}} \int_{-\delta}^{\delta} \mathbf{I}\Phi(\beta+iv)\, dv - $$

$$-\frac{1}{\sqrt{(2\pi)}} \int_{-\delta}^{\delta} \mathbf{I}\Phi(\alpha+iv)\, dv,$$

and it is sufficient to prove that the last two terms tend to zero as $\beta \to \infty$, $\alpha \to -\infty$ with δ fixed, say $\delta = 1$.

Let

$$\Omega(w) = \Phi(w) - \frac{i}{w\sqrt{(2\pi)}} \int_0^\infty \psi^2\, dx - \frac{i}{\sqrt{(2\pi)}} \sum_{\beta-1\leqslant w_n\leqslant\beta+1} \frac{c_n^2}{w-w_n}.$$

Then $\Omega(w)$ is regular for $\beta-1 \leqslant u \leqslant \beta+1$ (if $\beta > 1$). By (6.5) and the convergence of $\sum c_n^2$

$$|\Omega(w)| \leqslant \eta(u)/|v|,$$

where $\eta(u) \to 0$ as $u \to \infty$. Hence by Lemma α

$$|\Omega(\beta+iv)| \leqslant A\eta(\beta).$$

Hence

$$\int_{-1}^{1} \mathbf{I}\Phi(\beta+iv)\,dv = O\{\eta(\beta)\}+O\left(\frac{1}{\beta}\right)+$$

$$+\sum_{\beta-1\leqslant w_n\leqslant\beta+1} c_n^2 \int_{-1}^{1} \frac{u-w_n}{(u-w_n)^2+v^2}\,dv$$

$$= O\{\eta(\beta)\}+O(\beta^{-1})+O\Bigl(\sum_{\beta-1\leqslant w_n\leqslant\beta+1} c_n^2\Bigr) = o(1)$$

as $\beta \to \infty$, and similarly for the integral involving α. Hence (6.10) is an equality, i.e. the Parseval formula holds. The result is also equivalent to

$$\lim_{N\to\infty} \int_0^\infty \Bigl\{\psi(x) - \sum_{n=1}^{N} c_n \psi_n(x)\Bigr\}^2 dx = 0. \tag{6.11}$$

7. A familiar argument now shows that this result, proved for a restricted class of functions, actually holds for all functions of $L^2(0,\infty)$. For, if $\psi(x)$ is $L^2(0,\infty)$, we can define $\omega(x)$ satisfying the conditions imposed on $\psi(x)$ at the beginning of § 6, such that

$$\int_0^\infty \{\psi(x)-\omega(x)\}^2\,dx < \epsilon.$$

We can then find N such that

$$\int_0^\infty \Bigl\{\omega(x) - \sum_{n=1}^{N} d_n \psi_n(x)\Bigr\}^2 dx < \epsilon,$$

where the d_n are the 'Fourier coefficients' of $\omega(x)$. Hence

$$\int_0^\infty \Bigl\{\psi(x) - \sum_{n=1}^{N} d_n \psi_n(x)\Bigr\}^2 dx < 4\epsilon.$$

The left-hand side is

$$\int_0^\infty \psi^2\,dx + \sum_{n=1}^{N} d_n^2 - 2\sum_{n=1}^{N} c_n d_n \geqslant \int_0^\infty \psi^2\,dx - \sum_{n=1}^{N} c_n^2.$$

Hence

$$\int_0^\infty \Big\{\psi(x) - \sum_{n=1}^N c_n \psi_n(x)\Big\}^2 \, dx = \int_0^\infty \psi^2 \, dx - \sum_{n=1}^N c_n^2 < 4\epsilon,$$

and (6.11) follows.

8. The convergence theorem. Now let ψ again satisfy the conditions stated at the beginning of § 6, (6.2) being true for some given w. Then, by (6.3),

$$\psi(x) = \int_0^\infty G(x,y)\chi(y) \, dy, \tag{8.1}$$

where $\chi(y) = L\psi(y) - w\psi(y)$. Since χ is L^2,

$$\chi(y) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n=1}^N \gamma_n \psi_n(y),$$

where the γ_n are the 'Fourier coefficients' of $\chi(y)$. Hence

$$\lim_{N\to\infty} \int_0^\infty G(x,y)\Big\{\chi(y) - \sum_{n=1}^N \gamma_n \psi_n(y)\Big\} \, dy = 0,$$

i.e.
$$\psi(x) = \sum_{n=1}^\infty \gamma_n \int_0^\infty G(x,y)\psi_n(y) \, dy = \sum_{n=1}^\infty \frac{\gamma_n}{w_n - w} \psi_n(x), \tag{8.2}$$

by (5.6). Since
$$\sum \left| \frac{\gamma_n}{w_n - w} \right|^2 < K \sum |\gamma_n|^2 < \infty,$$

the series on the right also converges in mean, viz. to $\psi(x)$, since the limit and mean limit of a sequence are equal almost everywhere. Hence

$$c_m = \lim_{N\to\infty} \int_0^\infty \Big\{\sum_{n=1}^N \frac{\gamma_n}{w_n - w} \psi_n(x)\Big\}\psi_m(x) \, dx = \frac{\gamma_m}{w_m - w},$$

so that the expansion of $\psi(x)$ has the 'Fourier' form. The series is clearly uniformly convergent over any finite interval. It is also absolutely convergent, since $\sum |\gamma_n|^2$ is convergent, and so is

$$\sum \left| \frac{\psi_n(x)}{w_n - w} \right|^2$$

since $G(x,y)$ is L^2 (for a fixed x).

As a particular case consider the expansion of $\Psi_+(x,w)$, where now $\psi(x)$ is any function of L^2. We have

$$W\{\Psi_+(x,w), g(x,w)\} = -\frac{i}{\sqrt{(2\pi)}} \int_x^\infty g(y,w)\psi(y) \, dy \to 0.$$

Hence the conditions for a convergent expansion are satisfied, and by (5.9) the expansion is

$$\Psi_+(x,w) = \frac{i}{\sqrt{(2\pi)}} \sum_{n=1}^{\infty} \frac{c_n}{w-w_n} \psi_n(x). \tag{8.3}$$

9. We shall now show that *the above results are valid provided that*

$$\int_0^\infty \int_0^\infty |G(x,y,w')|^2\,dxdy < \infty \tag{9.1}$$

for some complex w'.

The function corresponding to $\Psi_+(x,w)$ for the finite interval $(0,a)$ is

$$\Psi_+(x,w,a) = -\frac{i}{\sqrt{(2\pi)}} \int_0^a G(x,y,w,a)\psi(y)\,dy,$$

where $$G(x,y,w,a) = \begin{cases} g(x,w,a)f(y,w) & (y<x), \\ f(x,w)g(y,w,a) & (y>x), \end{cases}$$

$g(x,w,a)$ being $F(x)+lf(x)$. Now $\Psi_+(x,w,a)$ is meromorphic, with poles $w_{n,a}$ independent of x; for* $G(x,y,w,a)$ can be expressed as the quotient of two integral functions, of which the denominator is independent of x and y. The eigenfunctions $\psi_n(x,a)$ for the finite interval satisfy

$$\psi_n(x,a) = (w_{n,a}-w) \int_0^a G(x,y,w,a)\psi_n(y,a)\,dy.$$

The 'Bessel's inequality' argument then gives

$$\int_0^a\int_0^a |G(x,y,w,a)|^2\,dxdy - \sum_{m=1}^{n} \frac{1}{|w_{m,a}-w|^2}$$
$$= \int_0^a\int_0^a \left| G(x,y,w,a) - \sum_{m=1}^{n} \frac{\psi_m(x,a)\psi_m(y,a)}{w_{m,a}-w}\right|^2 dxdy \geqslant 0.$$

Now

$$\int_0^a\int_0^a |G(x,y,w,a)|^2\,dxdy = 2\int_0^a |g(x,w,a)|^2\,dx \int_0^x |f(y,w)|^2\,dy$$
$$\leqslant 2\int_0^a |g(x,w)|^2\,dx \int_0^x |f(y,w)|^2\,dy + 2|l-l_1|^2\left(\int_0^a |f(y,w)|^2\,dy\right)^2,$$

since $$g(x,w,a) = g(x,w)+(l-l_1)f(x,w).$$

* See E. L. Ince, *Ordinary Differential Equations* (London, 1927), § 11.12.

If $w = w'$, the first term on the right is bounded, by (9.1), and so is the second, by (2.12). Hence

$$\sum_{m=1}^{n} \frac{1}{|w_{m,a}-w'|^2} \leqslant K,$$

where K is independent of n and a.

If $|w_{m,a}| \leqslant R$ for $m = 1, 2,..., n$, then it follows that

$$n \leqslant K(|w'|^2+R^2).$$

Hence
$$p(w,a) = (w-w_{1,a})...(w-w_{n,a})$$

is bounded in any finite region of the w-plane. We can therefore choose a sequence of values of a such that $p(w,a)$ tends uniformly to a limit $p(w)$ in the region. Hence $p(w)$ is an integral function. Let its zeros be w_1, w_2,.... If $|p(w)| \geqslant m$ on $|w| = \rho$, we can choose a so that $|p(w,a)-p(w)| < m$. Then by Rouché's theorem $p(w)$ and $p(w,a)$ have the same number of zeros in the circle. It follows that the number of zeros of $p(w)$ in the whole plane does not exceed $K(|w'|^2+R^2)$, i.e. $p(w)$ is a polynomial. Rouché's theorem also shows that the w_m are the limits of the $w_{m,a}$, n being ultimately constant (i.e. equal to the number of zeros of $p(w)$).

Also

$$\Psi_+(x,w)-\Psi_+(x,w,a) = -\frac{i}{\sqrt{(2\pi)}}\{g(x,w)-g(x,w,a)\} \int_0^x f(y,w)\psi(y)\,dy -$$
$$-\frac{i}{\sqrt{(2\pi)}} f(x,w) \int_x^a \{g(y,w)-g(y,w,a)\}\psi(y)\,dy -$$
$$-\frac{i}{\sqrt{(2\pi)}} f(x,w) \int_a^\infty g(y,w)\psi(y)\,dy.$$

The first and third terms clearly tend to zero as $a \to \infty$ through a sequence such that $l \to l_1$. The modulus of the integral in the second term is

$$\left|(l-l_1)\int_x^a f(y,w)\psi(y)\,dy\right| \leqslant |l-l_1|\left(\int_0^a |f(y,w)|^2\,dy \int_0^\infty \psi^2\,dy\right)^{\frac{1}{2}}$$

which also tends to zero, as in the proof of (2.13). Hence $\Psi_+(x,w,a)$ tends to $\Psi_+(x,w)$, and clearly uniformly in any closed domain entirely in the upper half-plane.

Now consider

$$(w-w_{1,a})...(w-w_{n,a})\Psi_+(x,w,a).$$

This is regular for $|w| \leqslant R$, bounded on $|w| = |w_{n,a}|$, and, as $a \to \infty$ through certain values, tends to

$$(w-w_1)...(w-w_n)\Psi_+(x,w)$$

uniformly in part of this region. Hence it tends to this limit uniformly in $|w| \leqslant |w_n|-\epsilon$. Applying this with arbitrarily large R, it follows that $\Psi_+(x,w)$ is regular except for poles w_1, w_2,..., and that

$$\sum_{m=1}^{\infty} \frac{1}{|w'-w_m|^2} \leqslant K.$$

This proves the theorem stated.

In particular (9.1) holds if we are in the limit-circle case for $w = w'$, i.e. if both $f(x,w')$ and $g(x,w')$ are $L^2(0,\infty)$.

As an example, consider the Bessel function expansion over an interval $(0,b)$, where 0 corresponds to the ∞ of the above analysis. The solutions are

$$x^{\frac{1}{2}}J_\nu\{x(-w)^{\frac{1}{2}}\}, \qquad x^{\frac{1}{2}}Y_\nu\{x(-w)^{\frac{1}{2}}\}.$$

We are in the limit-circle case if $0 \leqslant \nu < \frac{1}{2}$, and in the limit-point case if $\nu \geqslant \frac{1}{2}$. But

$$\int_0^b dx \int_x^b |x^{\frac{1}{2}}J_\nu\{x(-w)^{\frac{1}{2}}\}y^{\frac{1}{2}}Y_\nu\{y(-w)^{\frac{1}{2}}\}|^2\,dy$$
$$< K\int_0^b x^{1+2\nu}\,dx \int_x^b y^{1-2\nu}\,dy < \infty.$$

Hence we obtain a series expansion in either case.

10. We can also show* that, if we are in the limit-circle case for any complex w', then so we are for every complex w. For, if $f(x,w')$ is L^2, let

$$\phi(x) = f(x,w')+(w-w')\int_0^\infty G(x,y,w)f(y,w')\,dy.$$

It is at once verified that $(L-w)\phi = 0$, and that

$$\phi(0)\cos h+\phi'(0)\sin h = 0.$$

Also $\phi(x)$ is L^2, by (3.9) with $\psi(x)$ replaced by $f(x,w')$. Since h is arbitrary, it follows that every solution of $(L-w)\phi = 0$ is L^2.

* Satz 5 of Weyl's first paper.

A similar remark applies to (9.1). By the Parseval theorem

$$\int_0^\infty |G(x,y,w)|^2\,dy = \sum_{n=1}^\infty \left|\frac{\psi_n(x)}{w-w_n}\right|^2.$$

If (9.1) holds, then $\sum |w_n|^{-2}$ is convergent, and we can integrate again, giving

$$\int_0^\infty\int_0^\infty |G(x,y,w)|^2\,dxdy = \sum_{n=1}^\infty \frac{1}{|w-w_n|^2}.$$

Hence (9.1) holds with w instead of w'.

GENERAL TRANSFORMATIONS AND THE PARSEVAL THEOREM ✓

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1. Introduction

IT is well known that, if $f(x)$ belongs to the Lebesgue class $L^2(0,\infty)$, then the Fourier cosine transform

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \sin xy\, f(y)\, \frac{dy}{y}$$

exists almost everywhere, and also belongs to $L^2(0,\infty)$. Further,

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \sin xy\, g(y)\, \frac{dy}{y}$$

almost everywhere, and, if $f_1(x)$, $g_1(x)$ and $f_2(x)$, $g_2(x)$ are two such pairs of transforms, then

$$\int_0^\infty f_1(x) f_2(x)\, dx = \int_0^\infty g_1(x) g_2(x)\, dx.$$

The latter result is usually known as the 'Parseval theorem for Fourier cosine transforms'.*

In the present paper we discuss certain sub-classes of $L^2(0,\infty)$ which also have the property that transforms of functions of these classes belong to the same class, and we find that there exist corresponding modifications of the Parseval theorem. The results are extended to cover the general transformations of Fourier type given by Watson.†

The classes of functions considered here are of fundamental importance in the theory of summation formulae, and I have mentioned some particular cases of these classes in previous papers on that subject.‡

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937) (referred to as F.I.), Chapter 3.

† G. N. Watson, *Proc. London Math. Soc.* (2) 35 (1933), 156–99.

‡ A. P. Guinand, *Quart. J. of Math.* (Oxford), 9 (1938), 53–67, and 10 (1939), 104–18.

2. The class of functions S_k^2

Definition. *If $f(x)$ is the kth integral over the range (x,∞) of its kth derivative in the sense that there exists a function $f^{(k)}(x)$ such that*

$$f(x) = \frac{1}{\Gamma(k)} \int_x^\infty (t-x)^{k-1} f^{(k)}(t)\, dt \tag{2.1}$$

almost everywhere, and if further $x^k f^{(k)}(x)$ belongs to the Lebesgue class $L^p(0,\infty)$, then we say that $f(x)$ belongs to S_k^p.*

Now consider a function $f(x)$ belonging to S_k^2. The function $x^k f^{(k)}(x)$ belongs to $L^2(0,\infty)$, and consequently has a Mellin transform† $\mathfrak{F}_k(s)$ belonging to $L^2(-\infty,\infty)$ on the line $\sigma = \frac{1}{2}$ $(s = \sigma + it)$. Let us write

$$\mathfrak{F}_k(s) = \frac{\Gamma(s+k)}{\Gamma(s)} \mathfrak{F}(s),$$

and consider the function

$$\mathfrak{F}_r(s) = \frac{\Gamma(s+r)}{\Gamma(s)} \mathfrak{F}(s) = \frac{\Gamma(s+r)}{\Gamma(s+k)} \mathfrak{F}_k(s),$$

where $0 \leqslant r < k$. Now $\Gamma(s+r)/\Gamma(s+k)$ is bounded for all values of s on the line $\sigma = \frac{1}{2}$. Hence $\mathfrak{F}_r(s)$ also belongs to $L^2(-\infty,\infty)$ on $\sigma = \frac{1}{2}$, and has a Mellin transform $\phi(x)$ belonging to $L^2(0,\infty)$. It is easily verified that the following pairs of functions are also Mellin transforms of the same Lebesgue classes:

$$\left.\begin{matrix} 0 & (x<y) \\ x^{-k-1} & (x>y) \end{matrix}\right\}, \qquad \frac{y^{s-k-1}}{k+1-s},$$

$$\left.\begin{matrix} 0 & (x<y) \\ \dfrac{x^{-k-1}y^{r-k}(x-y)^{k-r}}{\Gamma(k-r+1)} & (x>y) \end{matrix}\right\}, \qquad \frac{\Gamma(1-s+r)}{\Gamma(2-s+k)} y^{s-k-1}.$$

Hence, by the Parseval theorem for Mellin transforms,

$$\begin{aligned}\int_y^\infty x^{-k-1}\phi(x)\, dx &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s+r)}{\Gamma(s+k)} \mathfrak{F}_k(s) \frac{y^{-s-k}}{s+k}\, ds \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}_k(s) \frac{\Gamma(s+r)}{\Gamma(s+k+1)} y^{-s-k}\, ds \\ &= \frac{y^{r-k}}{\Gamma(k-r+1)} \int_y^\infty f^{(k)}(x)(x-y)^{k-r} \frac{dx}{x}. \end{aligned} \tag{2.2}$$

* The function $f^{(k)}(x)$ differs from the kth derivative of $f(x)$ by a factor $(-1)^k$. This factor has no significance for the present purpose, and it is more convenient to omit it. Cf. W. L. Ferrar, *Proc. Roy. Soc. Edinburgh*, 48 (1927), 92–105.

† F.I. 94–5.

Now let us consider the integral

$$f^{(r)}(x) = \frac{1}{\Gamma(k-r)}\int_x^\infty (t-x)^{k-r-1}f^{(k)}(t)\,dt. \tag{2.3}$$

We have

$$\int_a^\infty |(t-x)^{k-r-1}f^{(k)}(t)|\,dt \leqslant \left\{\int_a^\infty (t-x)^{2k-2r-2}t^{-2k}\,dt\right\}^{\frac{1}{2}}\left\{\int_a^\infty |t^kf^{(k)}(t)|^2\,dt\right\}^{\frac{1}{2}}.$$

Hence the integral (2.3) converges absolutely if $0 \leqslant r < k-\frac{1}{2}$. For $k-\frac{1}{2} \leqslant r < k$ the integral over the range (a,∞) where $a > x$ is absolutely convergent, and, if we put

$$\psi(x) = \int_x^a (t-x)^{k-r-1}f^{(k)}(t)\,dt,$$

then

$$\int_y^a \psi(x)\,dx = \int_y^a dx \int_x^a (t-x)^{k-r-1}f^{(k)}(t)\,dt$$

$$= \int_y^a f^{(k)}(t)\,dt \int_y^t (t-x)^{k-r-1}\,dx$$

$$= \frac{1}{k-r}\int_y^a (t-y)^{k-r}f^{(k)}(t)\,dt,$$

and the existence of any one of these integrals implies the existence and equality of the others. The last integral obviously does exist; hence $\psi(x)$ exists almost everywhere, and so does $f^{(r)}(x)$. Hence

$$\int_y^\infty x^{-k-1+r}f^{(r)}(x)\,dx = \frac{1}{\Gamma(k-r)}\int_y^\infty x^{-k-1+r}\,dx \int_x^\infty (t-x)^{k-r-1}f^{(k)}(t)\,dt$$

$$= \frac{1}{\Gamma(k-r)}\int_y^\infty f^{(k)}(t)\,dt \int_y^t x^{-k-1+r}(t-x)^{k-r-1}\,dx$$

$$= \frac{y^{r-k}}{\Gamma(k-r+1)}\int_y^\infty f^{(k)}(t)(t-y)^{k-r}\,\frac{dt}{t},$$

where the inversion is justified by absolute convergence. By (2.2)

$$\int_y^\infty x^{-k-1}\phi(x)\,dx = \int_y^\infty x^{-k-1+r}f^{(r)}(x)\,dx.$$

Thus
$$\phi(x) = x^r f^{(r)}(x) \tag{2.4}$$
almost everywhere, and we have proved

THEOREM 1. *If $f(x)$ belongs to S_k^2, and $0 \leqslant r < k$, then $f^{(r)}(x)$, defined by (2.3), exists almost everywhere, and $x^r f^{(r)}(x)$ belongs to $L^2(0,\infty)$; in particular $f(x) = f^{(0)}(x)$ belongs to $L^2(0,\infty)$. If $\mathfrak{F}(s)$ is the Mellin transform of $f(x)$, then*
$$\frac{\Gamma(s+r)}{\Gamma(s)}\mathfrak{F}(s)$$
is the Mellin transform of $x^r f^{(r)}(x)$.

An immediate corollary of Theorem 1 is

THEOREM 2. *If $f(x)$ belongs to S_k^2, then it belongs to S_r^2 for any r in $0 \leqslant r \leqslant k$. In particular it belongs to $L^2(0,\infty)$.*

To prove this we need only show that
$$f(x) = \frac{1}{\Gamma(r)} \int\limits_x^\infty (t-x)^{r-1} f^{(r)}(t)\, dt$$
almost everywhere. This can be proved in the same way as (2.4).

3. General transforms and the Parseval theorem for functions of S_k^2

Watson* proved the following results on general transformations of the Fourier type:

THEOREM 3. *If $\mathfrak{K}(s)$ is defined for $\sigma = R(s) = \frac{1}{2}$, and*
$$\mathfrak{K}(\tfrac{1}{2}+it)\mathfrak{K}(\tfrac{1}{2}-it) = |\mathfrak{K}(\tfrac{1}{2}+it)|^2 = 1,$$
then
$$\frac{k_1(x)}{x} = \frac{1}{2\pi i} \operatorname*{l.i.m.}_{T\to\infty} \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \mathfrak{K}(s) \frac{x^{-s}}{1-s}\, ds$$
exists, and belongs to $L^2(0,\infty)$. Further, if $f(x)$ belongs to $L^2(0,\infty)$, then
$$g(x) = \frac{d}{dx} \int\limits_0^\infty k_1(xu) f(u) \frac{du}{u}$$
defines almost everywhere a function $g(x)$, also belonging to $L^2(0,\infty)$. The reciprocal formula
$$f(x) = \frac{d}{dx} \int\limits_0^\infty k_1(xu) g(u) \frac{du}{u}$$

* F.I. Chapter 8.

also holds almost everywhere, and, if $f_1(x)$, $g_1(x)$ and $f_2(x)$, $g_2(x)$ are two such pairs of transforms, then

$$\int_0^\infty f_1(x)f_2(x)\,dx = \int_0^\infty g_1(x)g_2(x)\,dx.$$

If $\mathfrak{F}(s)$ and $\mathfrak{G}(s)$ are the Mellin transforms of $f(x)$ and $g(x)$ respectively, then

$$\mathfrak{F}(s) = \mathfrak{K}(s)\mathfrak{G}(1-s),$$

$$\mathfrak{G}(s) = \mathfrak{K}(s)\mathfrak{F}(1-s)$$

almost everywhere on the line $\sigma = \frac{1}{2}$.

From Theorems 1 and 3 we can deduce the following result:

THEOREM 4. *If $k_1(x)$ is a Fourier kernel in the sense of Theorem 3, and $f(x)$ belongs to S_k^2, then*

$$g(x) = \frac{d}{dx}\int_0^\infty k_1(xu)f(u)\,\frac{du}{u}$$

defines almost everywhere a function $g(x)$, also belonging to S_k^2, and the reciprocal relation

$$f(x) = \frac{d}{dx}\int_0^\infty k_1(xu)g(u)\,\frac{du}{u}$$

also holds almost everywhere.

The reciprocal relationship follows immediately from Theorem 3, since $f(x)$ belongs to $L^2(0,\infty)$. By Theorem 1 the Mellin transform of $x^k f^{(k)}(x)$ is

$$\frac{\Gamma(s+k)}{\Gamma(s)}\mathfrak{F}(s),$$

and this belongs to $L^2(-\infty,\infty)$ on $\sigma = \frac{1}{2}$. Also

$$\left|\frac{\Gamma(s+k)}{\Gamma(s)}\mathfrak{G}(s)\right| = \left|\frac{\Gamma(s+k)}{\Gamma(s)}\mathfrak{K}(s)\mathfrak{F}(1-s)\right|$$

$$= \left|\frac{\Gamma(1-s)\Gamma(s+k)}{\Gamma(s)\Gamma(1-s+k)}\right|\left|\mathfrak{K}(s)\right|\left|\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\mathfrak{F}(1-s)\right|$$

$$= \left|\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\mathfrak{F}(1-s)\right|$$

on $\sigma = \frac{1}{2}$. Hence

$$\frac{\Gamma(s+k)}{\Gamma(s)}\mathfrak{G}(s)$$

also belongs to $L^2(-\infty,\infty)$ on $\sigma=\frac{1}{2}$, and has a Mellin transform $x^k g^{(k)}(x)$ belonging to $L^2(0,\infty)$. As before, it follows that the function

$$\frac{1}{\Gamma(k)}\int_x^\infty (t-x)^{k-1}g^{(k)}(t)\,dt$$

exists and is equal to $g(x)$ almost everywhere. The theorem now follows from the definition of S_k^2.

We can also prove an extended form of the Parseval theorem for this type of transform.

THEOREM 5. *If $f_1(x)$, $g_1(x)$ and $f_2(x)$, $g_2(x)$ are two pairs of transforms of the type of Theorem 4, and $0\leqslant\alpha\leqslant k$, $0\leqslant\beta\leqslant k$, then*

$$\int_0^\infty x^{\alpha+\beta}f_1^{(\alpha)}(x)f_2^{(\beta)}(x)\,dx=\int_0^\infty x^{\alpha+\beta}g_2^{(\alpha)}(x)g_1^{(\beta)}(x)\,dx,$$

where the functions involved are defined as in Theorem 1.

By Theorem 1 the Mellin transforms of $x^\alpha f_1^{(\alpha)}(x)$, $x^\beta f_2^{(\beta)}(x)$, $x^\alpha g_2^{(\alpha)}(x)$, $x^\beta g_1^{(\beta)}(x)$ are

$$\frac{\Gamma(s+\alpha)}{\Gamma(s)}\mathfrak{F}_1(s),\qquad \frac{\Gamma(s+\beta)}{\Gamma(s)}\mathfrak{F}_2(s),\qquad \frac{\Gamma(s+\alpha)}{\Gamma(s)}\mathfrak{G}_2(s),\qquad \frac{\Gamma(s+\beta)}{\Gamma(s)}\mathfrak{G}_1(s),$$

in an obvious notation. Hence, by the Parseval theorem for Mellin transforms and Theorem 3,

$$\int_0^\infty x^{\alpha+\beta}f_1^{(\alpha)}(x)f_2^{(\beta)}(x)\,dx$$

$$=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(s+\alpha)}{\Gamma(s)}\mathfrak{F}_1(s)\frac{\Gamma(1-s+\beta)}{\Gamma(1-s)}\mathfrak{F}_2(1-s)\,ds$$

$$=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(s+\alpha)}{\Gamma(s)}\mathfrak{K}(s)\mathfrak{G}_1(1-s)\frac{\Gamma(1-s+\beta)}{\Gamma(1-s)}\mathfrak{K}(1-s)\mathfrak{G}_2(s)\,ds$$

$$=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(s+\alpha)}{\Gamma(s)}\mathfrak{G}_2(s)\frac{\Gamma(1-s+\beta)}{\Gamma(1-s)}\mathfrak{G}_1(1-s)\,ds$$

$$=\int_0^\infty x^{\alpha+\beta}g_2^{(\alpha)}(x)g_1^{(\beta)}(x)\,dx,$$

as required.

A NOTE ON THE SUMMABILITY OF LACUNARY PARTIAL SUMS OF FOURIER SERIES

By FU-TRAING WANG (*Kweichow*)

[Received 11 February 1941]

LET $f(t)$ be an integrable periodic function with period 2π, and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Put
$$S_n = S_n(t) = \tfrac{1}{2}a_0 + \sum_{p=1}^{n} (a_p \cos pt + b_p \sin pt)$$

and
$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\};$$

then we know that a Lebesgue integrable function exists such that the lacunary partial sum $S_{n^2}(x)$ diverges almost everywhere.* Z. Zalcwasser† has proved that $S_{n^2}(x) \to f(x)$ $(C, 1)$ almost everywhere, and has asked whether the summability $(C, 1)$ can be replaced by (C, α). To this the answer is affirmative, as the following theorem shows.

THEOREM. *If*
$$\int_0^x |\phi(t)|\, dt = o(x),$$

then
$$S_{n^2}(x) \to f(x)\ (C, \alpha) \quad (0 < \alpha < 1), \quad \textit{as } n \to \infty.$$

Put
$$\binom{n+\alpha}{n}\sigma_n = \sum_{p=0}^{n} \binom{p+\alpha-1}{p} S_{(n-p)^2} - f(x) = \frac{1}{\pi}\int_0^{\pi} \phi(t)\frac{H_n(t)}{2\sin\frac{1}{2}t}\, dt,$$

where
$$H_n(t) = \sum_{p=0}^{n} \binom{p+\alpha-1}{p} \sin\{(n-p)^2+\tfrac{1}{2}\}t.$$

It suffices to prove that $\sigma_n \to 0$ as $n \to 0$. We require the following two lemmas.

Put
$$J_n(t) = \sum_{p=0}^{n} \binom{p+\alpha-1}{p} e^{i(n-p)^2 t},$$

then
$$|H_n(t)| \leqslant |J_n(t)|.$$

* A. Kolmogoroff, *Fundamenta Math.* 4 (1914).

† Z. Zalcwasser, *Studia Math.* 6 (1936).

LEMMA 1.

$$J_n(t) = O(n^\alpha t^{\frac{1}{2}}) + O\{t^{-\frac{1}{2}} n^{\alpha-1} (\log n)^{1/\alpha - 1}\} + O\left(\frac{n^\alpha}{\log n}\right).$$

Proof. $$J_n(t) = \sum_{p=0}^{n-1} E_p \Delta\binom{p+\alpha-1}{p} + \binom{\alpha-1+n}{n} E_n,$$

where $$E_p = \sum_{\nu=0}^{p} e^{i(n-\nu)^2 t}.$$

By van der Corput's theorem,* we have

$$E_p = O(pt^{\frac{1}{2}}) + O(t^{-\frac{1}{2}}).$$

Put $r_n = [n(\log n)^{-1/\alpha}]$. Then

$$J_n(t) = \sum_{p=r_n}^{n-1} + \sum_{p=0}^{r_n-1} + O(n^\alpha t^{\frac{1}{2}}) + O(n^{\alpha-1} t^{-\frac{1}{2}})$$

$$= \sum_{p=r_n}^{n-1} O(pt^{\frac{1}{2}}) \Delta\binom{p+\alpha-1}{p} + \sum_{p=r_n}^{n-1} O(t^{-\frac{1}{2}}) \Delta\binom{p+\alpha-1}{p} +$$

$$+ O(n^\alpha t^{\frac{1}{2}}) + O(n^{\alpha-1} t^{-\frac{1}{2}}) + O\left(\frac{n^\alpha}{\log n}\right)$$

$$= O\left\{\binom{r_n+\alpha-1}{r_n} t^{-\frac{1}{2}}\right\} + O(n^\alpha t^{\frac{1}{2}}) + O\left(\frac{n^\alpha}{\log n}\right).$$

Thus the lemma is proved.

LEMMA 2.

$$J_n(t) = O(n^{-\alpha} t^{-\alpha}) + O(n^{\alpha-1} t^{-\frac{1}{2}}) + O\left(\frac{n^\alpha}{\log n}\right)$$

for $n^{-2} \leqslant t \leqslant n^{-1}$.

Proof. Now let $p_n = [n(\log n)^{-1}]$. Then

$$J_n(t) = \sum_{p=p_n}^{n-r_n} \binom{n-p+\alpha-1}{n-p} e^{ip^2 t} +$$

$$+ \sum_{p=0}^{p_n-1} \binom{n-p+\alpha-1}{n-p} e^{ip^2 t} + \sum_{p=n-r_n+1}^{n} \binom{n-p+\alpha-1}{n-p} e^{ip^2 t}.$$

By Stirling's formula we have

$$J_n(t) = \frac{1}{\Gamma(\alpha)} \sum_{p=p_n}^{n-r_n} (n-p)^{\alpha-1} e^{ip^2 t} + O\{n^\alpha (\log n)^{-1}\}.$$

* E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 2 (1931), 313–20.

By a method due to van der Corput* we can approximate the finite sum to an integral. Put $a = p_n - 1$, $b = n - r_n + 1$, $f(x) = x^2 t/2\pi$; then $0 < f'(x) < \frac{1}{2}$, and we have, if $\chi(x) = x - [x] - \frac{1}{2}$,

$$2\pi i \int_a^b \chi(x)(n-x)^{\alpha-1} e^{2\pi i f(x)} f'(x)\,dx$$

$$= -\sum_{\nu=1}^{\infty} \frac{2i}{\nu} \int_a^b (n-x)^{\alpha-1} \sin 2\pi\nu x\, e^{2\pi i f(x)} f'(x)\,dx$$

$$= \sum_{\nu=1}^{\infty} \frac{1}{\nu} O\left(r_n^{\alpha-1} \frac{1}{2\nu-1}\right) = O\{n^{\alpha-1}(\log n)^{1/\alpha-1}\}.$$

Then

$$2\pi i \int_a^b (x-\tfrac{1}{2})(n-x)^{\alpha-1} e^{2\pi i f(x)} f'(x)\,dx$$

$$= e^{2\pi i f(b)}(b-\tfrac{1}{2})(n-b)^{\alpha-1} - e^{2\pi i f(a)}(a-\tfrac{1}{2})(n-a)^{\alpha-1} -$$

$$- \int_a^b (n-x)^{\alpha-1} e^{2\pi i f(x)}\,dx - (1-\alpha)\int_a^b (x-\tfrac{1}{2})(n-x)^{\alpha-2} e^{2\pi i f(x)}\,dx.$$

But

$$2\pi i \int_a^b [x](n-x)^{\alpha-1} e^{2\pi i f(x)} f'(x)\,dx = \sum_{p=a}^{b-1} 2\pi i p \int_p^{p+1} (n-x)^{\alpha-1} e^{2\pi i f(x)} f'(x)\,dx$$

$$= (b-1)(n-b)^{\alpha-1} e^{2\pi i f(b)} - a(n-a)^{\alpha-1} e^{2\pi i f(a)} -$$

$$- \sum_{n=a+1}^{b-1} (n-p)^{\alpha-1} e^{2\pi i f(p)} - (1-\alpha)\int_a^b [x](n-x)^{\alpha-2} e^{2\pi i f(x)}\,dx.$$

Hence

$$J_n(t) = \frac{1}{\Gamma(\alpha)} \int_a^b (n-x)^{\alpha-1} e^{2\pi i f(x)}\,dx + \tfrac{1}{2}(n-a)^{\alpha-1} e^{2\pi i f(a)} -$$

$$-\tfrac{1}{2}(n-b)^{\alpha-1} e^{2\pi i f(b)} - (1-\alpha)\int_a^b (x-[x]-\tfrac{1}{2})(n-x)^{\alpha-2} e^{2\pi i f(x)}\,dx +$$

$$+ O\left(\frac{n^\alpha}{\log n}\right)$$

$$= \frac{1}{\Gamma(\alpha)} \int^{\delta n} (n-x)^{\alpha-1} e^{ix^2 t}\,dx + \frac{1}{\Gamma(\alpha)} \int_{\delta n}^{n} (n-x)^{\alpha-1} e^{ix^2 t}\,dx + O\left(\frac{n^\alpha}{\log n}\right)$$

† See E. C. Titchmarsh, loc. cit.

$$= \frac{1}{2\Gamma(\alpha)\sqrt{t}} \int\limits_{a^2 t}^{\delta^2 n^2 t} \left(n - \sqrt{\frac{u}{t}}\right)^{\alpha-1} \frac{e^{iu}}{\sqrt{u}}\, du + \frac{n^\alpha}{\Gamma(\alpha)} \int\limits_{\delta}^{1} (1-x)^{\alpha-1} e^{ix^2 n^2 t}\, dx + {}$$

$$+ O\left(\frac{n^\alpha}{\log n}\right)$$

$$= O(n^{\alpha-1} t^{-\frac{1}{2}}) + \frac{n^\alpha}{\Gamma(\alpha)} \int\limits_{0}^{1-\delta} x^{\alpha-1} e^{i(1-x)^2 n^2 t}\, dx + O\left(\frac{n^\alpha}{\log n}\right)$$

$$= O(n^{\alpha-1} t^{-\frac{1}{2}}) +$$

$$+ \frac{1}{\Gamma(\alpha) n^\alpha t^\alpha} \int\limits_{0}^{(1-\delta)n^2 t} x^{\alpha-1} \exp\left\{i\left(1 - \frac{x}{n^2 t}\right)^2 n^2 t\right\} dx + O\left(\frac{n^\alpha}{\log n}\right)$$

$$= O(n^{\alpha-1} t^{-\frac{1}{2}}) + O\left\{n^{-\alpha} t^{-\alpha}\left(\int\limits_{0}^{1} + \int\limits_{1}^{(1-\delta)n^2 t}\right)\right\} + O\left(\frac{n^\alpha}{\log n}\right)$$

$$= O(n^{\alpha-1} t^{-\frac{1}{2}}) + O(n^{-\alpha} t^{-\alpha}) + O\left(\frac{n^\alpha}{\log n}\right).$$

Proof of the Theorem.

$$\sigma_n = O\left\{\frac{1}{n^\alpha}\left(\int\limits_{0}^{n^{-2}} + \int\limits_{n^{-2}}^{n^{-2}(\log n)^{(2/\alpha)-2}} + \right.\right.$$

$$\left.\left. + \int\limits_{n^{-2}(\log n)^{(2/\alpha)-2}}^{n^{-1}(\log n)^{(1/\alpha)-1}} + \int\limits_{n^{-1}(\log n)^{(1/\alpha)-1}}^{\delta} + \int\limits_{\delta}^{\pi}\right)\right\} = o(1).$$

The first and the last terms are estimated just like those of Zalcwasser's work, the second by Lemma 2, and the third and the fourth by Lemma 1.

SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS

By T. W. CHAUNDY (*Oxford*)

[Received 25 December 1940]

1. A symbolic form

THE condition

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

for the integrability of the total differential equation

$$P\,dx+Q\,dy+R\,dz=0$$

can conveniently be written in the symbolic form

$$\begin{vmatrix} P & \partial/\partial x & P \\ Q & \partial/\partial y & Q \\ R & \partial/\partial z & R \end{vmatrix}=0, \tag{1}$$

where the differential operators act, as usual, to the right. This symbolic form extends readily to *systems* of total differential equations

$$\sum_{s=1}^{n} P_{rs}\,dx_s=0 \quad (r=1,\ldots,m).$$

If $n=m+2$, we shall have the m conditions of integrability

$$|P_{1s} \quad P_{2s} \quad \ldots \quad P_{ms} \quad \partial/\partial x_s \quad P_{rs}|=0 \quad (r=1,\ldots,m). \tag{2}$$

If $n>m+2$, we must replace (2), interchanging rows and columns, by the m systems of equations

$$\left\|\begin{matrix} P_{1s} \\ \vdots \\ P_{ms} \\ \partial/\partial x_s \\ P_{rs} \end{matrix}\right\|=0 \quad (r=1,\ldots,m), \tag{3}$$

the differential operators acting downwards.

In this form it is simple to show that the conditions are (i) necessary, (ii) invariant, and therefore (iii) sufficient. I illustrate this by considering the pair of equations

$$\left.\begin{aligned} P_1\,dx_1+P_2\,dx_2+P_3\,dx_3+P_4\,dx_4=0 \\ Q_1\,dx_1+Q_2\,dx_2+Q_3\,dx_3+Q_4\,dx_4=0 \end{aligned}\right\}. \tag{4}$$

2. Conditions necessary

If this pair of equations is equivalent to the pair of exact equations $du = 0$, $du' = 0$, there must be two pairs of multipliers (λ, μ), (λ', μ') such that

$$\left.\begin{aligned} \lambda P_s + \mu Q_s &= \frac{\partial u}{\partial x_s} \\ \lambda' P_s + \mu' Q_s &= \frac{\partial u'}{\partial x_s} \end{aligned}\right\} \quad (s = 1, 2, 3, 4). \tag{5}$$

Thus evidently

$$|P_s \quad Q_s \quad \partial/\partial x_s \quad \lambda P_s + \mu Q_s| = 0,$$

i.e.

$$\lambda \Delta_1 + \mu \Delta_2 = 0,$$

where

$$\Delta_1 = |P_s \quad Q_s \quad \partial/\partial x_s \quad P_s|, \qquad \Delta_2 = |P_s \quad Q_s \quad \partial/\partial x_s \quad Q_s|;$$

and, similarly,

$$\lambda' \Delta_1 + \mu' \Delta_2 = 0.$$

But, if

$$\begin{vmatrix} \lambda & \mu \\ \lambda' & \mu' \end{vmatrix} = 0,$$

then (5) gives

$$\left\| \begin{matrix} \partial u/\partial x_s \\ \partial u'/\partial x_s \end{matrix} \right\| = 0, \quad \text{i.e. } f(u, u') = 0.$$

This we exclude since the exact equations $du = 0$, $du' = 0$ must be independent. Hence $\Delta_1, \Delta_2 = 0$, which are the conditions (2).

3. Conditions invariant

The above analysis shows that the conditions (2) are invariant for any linear combination of (4)

$$\sum (\lambda P_s + \mu Q_s)\, dx_s = 0, \qquad \sum (\lambda' P_s + \mu' Q_s)\, dx_s = 0.$$

If we change the variables (x_s) to (y_s) so that the equations become

$$\sum P'_s\, dy_s = 0, \qquad \sum Q'_s\, dy_s = 0,$$

where

$$P'_s = \sum_{t=1}^{4} \frac{\partial x_t}{\partial y_s} P_t, \qquad Q'_s = \sum_{t=1}^{4} \frac{\partial x_t}{\partial y_s} Q_t, \tag{6}$$

the conditions become

$$\left| P'_s \quad Q'_s \quad \frac{\partial}{\partial y_s} \quad \sum P_t \frac{\partial x_t}{\partial y_s} \right| = 0 \tag{7}$$

and the similar condition in Q.

In the differentiation of the last column in (7) the second factors of the products give rise to

$$\sum_t P_t \begin{vmatrix} P'_s & Q'_s & \dfrac{\partial}{\partial y_s} & \dfrac{\partial x_t}{\partial y_s} \end{vmatrix},$$

which vanishes. We can thus treat these second factors as constant coefficients. Substituting for P'_s, Q'_s and putting

$$\frac{\partial}{\partial y_s} = \sum_{t=1}^{4} \frac{\partial x_t}{\partial y_s} \frac{\partial}{\partial x_t},$$

we rewrite (7) as

$$\frac{\partial(x_1,\dots,x_4)}{\partial(y_1,\dots,y_4)} \times |P_s \quad Q_s \quad \partial/\partial x_s \quad P_s| = 0,$$

which shows that the conditions are invariant for change of variable.

4. Conditions sufficient

To show that the conditions are sufficient make x_4 (say) constant. The equations then reduce to the pair of ordinary equations

$$\frac{dx_1}{P_2 Q_3 - P_3 Q_2} = \frac{dx_2}{P_3 Q_1 - P_1 Q_3} = \frac{dx_3}{P_1 Q_2 - P_2 Q_1},$$

which we may suppose to have a solution of the form

$$u(x_1, x_2, x_3, x_4) = C, \qquad u'(x_1, x_2, x_3, x_4) = C',$$

when x_4 is constant. Thus, differentiating totally, we see that the given system (4) is equivalent to

$$du + \theta\, dx_4 = 0, \qquad du' + \theta'\, dx_4 = 0 \tag{8}$$

for some θ, θ'. Changing the variables to u, u', x_3, x_4 we can apply the conditions (2), since they are invariant, to the forms (8); it will be found that they reduce to

$$\frac{\partial \theta}{\partial x_3} = 0, \qquad \frac{\partial \theta'}{\partial x_3} = 0.$$

Thus θ, θ' are functions of u, u', x_4 only, and the pair of equations (4) has reduced to the pair of ordinary equations

$$\frac{du}{\theta} = \frac{du'}{\theta'} = -dx_4,$$

which we may suppose integrable.

The symbolic form (1) is given by Tanner,* who generalizes it in a different form for a different purpose—to discuss the reduction of a single total differential expression to a form in fewer variables. I have not seen a reference to it in the recent literature, but, even if the results are not new, it is perhaps worth while to recall them.

* H. W. L. Tanner, *Quart. J. of Math.* 16 (1879), 45–57 (46).